Confluence of Shallow Right-Linear Systems

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Rewrite Systems

Define a binary relation over a set of terms

Two main interpretations:

- Model of some dynamical system:

set of terms	$\mapsto$	state space
rewrite relation	$\mapsto$	dynamics

- Defining an equational theory

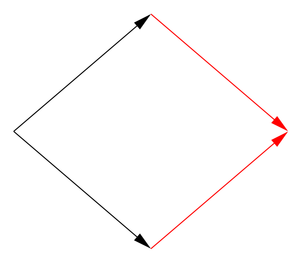
set of terms	$\mapsto$	elements in the model of the theory
rewrite relation	$\mapsto$	equational identities in the theory: simplification

Confluence

Two main properties of rewrite systems: confluence and termination

Confluence: Interpretations–

- Model of some dynamical system: a general definition of determinism
- Equational reasoning: decide word problem, assuming termination



Known Results

Class	Reachability	Confluence	Comment
General TRS	undecidable	undecidable	Turing-complete
Shallow TRS	undecidable	?	[Jacquemard 2003]
Linear TRS	undecidable	?	
RL-FPO TRS	decidable	?	[Takai, Kaji, Seki 2000]
Shallow RL	decidable	decidable	This work
Shallow Linear	decidable	decidable	[Godoy, T, Verma 2003]
Ground TRS	decidable	decidable	PTime [Godoy+ 00, T 01]

Shallow Right Linear Rewrite Systems

- Shallow: All variables occur at depth at most one
- Right Linear: Variables are not repeated on the RHS terms

Example of a shallow right-linear rewrite system:

$$R = \{x \vee x \to x,\ x \vee y \to y \vee x,\ x \vee 0 \to x,\ x \vee 1 \to 1\}.$$

Results for shallow right-linear systems:

- Word problem is decidable [Comon+94, Niu96]
- Reachability and joinability are decidable [Takai+00]

Approach

R is confluent if

$$\forall s, t : s \leftrightarrow_R^* t \;\Rightarrow\; \exists u . s \rightarrow_R^* u \leftarrow_R^* t$$

Instead of checking for all s, t, we reduce the check to terms s, t from a finite set, but with respect to a slightly modified $\overline{R}$.

Key Idea 1: Finite set consists of constants, variables and top-stable flat terms

Top Stable Terms

A term is top-stable if it cannot be rewritten to a constant/variable.

Example: $x \vee y$ is top-stable, whereas $x \vee 1$ is not.

Why are top-stable terms important for confluence?

If s, t are not top-stable, then $\exists \alpha, \beta$ s.t.

$$s \rightarrow_R^* \alpha, \quad t \rightarrow_R^* \beta.$$

If s, t are equivalent, then so are α, β. But we explicitly check for joinability of all equivalent α, β.

Problem: There are infinitely many top-stable terms.

Top Stabilizable Constants

A constant that is R-equivalent to a top-stable term.

Why are top-stabilizable constants important for confluence?

Sup. c is top-stabilizable and top-stable s is equivalent to c:

Given	Need to check
$u[c] \leftrightarrow^* v$	$u[c] \downarrow v$
$u[s] \leftrightarrow^* v$	$u[s] \downarrow v$
$u[\overline{c}] \leftrightarrow^*_{\overline{R}} v$	$u[\overline{c}] \downarrow_{\overline{R}} v$

Top-stabilizable c should not be used in the joinability proof. Why?

Shallow Right-Linear TRSs

Confluence preserving transformation:

$$\text{Shallow right-linear} \quad \mapsto \quad \text{Flat right-linear}$$

Flat TRSs can only use depth zero terms or non-top-stable terms in rewrite derivations.

Example. Unused subterms can be generalized:

$$\begin{array}{llllllll} (x \vee y) \vee \underline{(z \vee 1)} & \to & (x \vee y) \vee \underline{1} & \to & x \vee y & \to & y \vee x \\ w \vee \underline{(z \vee 1)} & \to & w \vee \underline{1} & \to & w & \to^* & w \end{array}$$

Example: Useless positions can be generalized:

$$1 \;\to\; (x \vee y) \vee 1 \quad \mapsto \quad 1 \;\to\; z \vee 1$$

Extending R to $\overline{R}$

Fixpoint computation: Incrementally add $\overline{c}$ for constants c that can be detected to be top-stabilizable.

$$
\begin{aligned}
R_0 &= R \\
R_{i+1} &= R_i \cup \{c \rightarrow \overline{d} : c, d \in \Sigma_0, \exists \text{ flat term } t \in \mathcal{T}(\Sigma \cup \overline{\Sigma}_0, \mathcal{V}) : \\
&\qquad\qquad t \leftrightarrow^*_{R_i} c \leftrightarrow^*_R d,\ t \text{ is top-stable wrt } R_i\} \\
\overline{R} &= \bigcup_i R_i
\end{aligned}
$$

If $c \rightarrow \overline{d} \in \overline{R}$, then d is top-stabilizable.

Detecting Top-Stabilizable Constants

Key Idea 2: The detection of top-stabilizable constants is related to the "confluentness" of R.

Let R be confluent upto height h:
—i.e., any set of equivalent terms with height $\leq h$ is joinable.

Then, if t is a top-stable term with height $\leq h+1$ and equivalent to c, then t is detected:
—i.e., $c \rightarrow \overline{d} \in \overline{R}$ for some $\overline{d} \in \overline{\Sigma_0}$.

Confluence Characterization

R is confluent iff the following two conditions hold:

(i) Every R-equivalent set of constants is R-joinable.

(ii) Let $\{\alpha_1, \ldots, \alpha_k, t_1, \ldots, t_n\}$ be an $\overline{R}$-equivalent set of terms, where
– $\alpha_i \in \Sigma_0 \cup \mathcal{V}$, and
– t_i are top-stable flat terms wrt $\overline{R}$.

Then, $\exists t'_1, \ldots, t'_n$ s.t.
– every t'_i is either t_i or $\overline{c}$ or x,
– some t'_i coincides with t_i, and
– the set $\{\alpha_1, \ldots, \alpha_k, t'_1, \ldots, t'_n\}$ is $\overline{R}$-joinable.

Example

$R = \{x \vee x \to x,\ x \vee y \to y \vee x,\ x \vee 0 \to x,\ x \vee 1 \to 1\}$.

- $0 \not\leftrightarrow^* 1$ and $0 \not\leftrightarrow^* x$ and $1 \not\leftrightarrow^* x$.
  Hence, condition (i) is vacuously true.
- Any term equivalent to 0 rewrites to 0. Same for 1 and x.
  Hence, none of 0, 1, and x are top-stabilizable.
  $\therefore \overline{R} = R$.
- The set $\{x \vee y, y \vee x\}$ is the only equivalent set of flat top-stable terms.
  But, this is joinable.
  Hence, condition (ii) is also true.
- Hence, R is confluent.

Proof Idea

$\Leftarrow$: If conditions (i) and (ii) are true, then
– R is confluent and
– all top-stabilizable constants are detected.

- Pick the minimal witness; it is either witness for
  (a) to nondetection of top-stable term.
  (b) to nonconfluence, or
- If (a), then we can get a smaller witness for nonconfluence. $\perp$.
- If (b), then it can be mapped to a set of the form covered by condition (ii).

$\Rightarrow$: Project derivation of R-joinability onto $\overline{R}$-joinability over flat terms.

Main Result

Confluence of shallow and right-linear term rewrite systems is decidable.

- Flatten R into a flat right-linear system
- Detect top-stabilizable constants and construct $\overline{R}$
- Check all equivalent constants are R-joinable
- Compute all sets of equivalent flat top-stable terms. Test if they are joinable, according to condition (ii)
- All above steps are possible because equivalence, reachability, and joinability are decidable for R and $\overline{R}$

Reflections

- Decidability of confluence for shallow right-linear systems is very surprising
- Proof is technical, but the high-level proof is similar to those for the special cases
  Each generalization is getting exponentially harder
- Crucial points for confluence of a TRS class:
  - Is equivalence decidable?
  - Reachability and joinability used as black-box?
  - Is the class asymmetric?
- Open problem: Confluence of RL-FPO systems.