Notes on Confluence of Ultra-Weakly-Left-Linear SDCTRSs via a Structure-Preserving Transformation

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Oriented Conditional Term Rewriting System (CTRS)

- A CTRS is a set $\mathcal{R}$ of oriented conditional rules

$$\ell \to r \Leftarrow s_1 \twoheadrightarrow t_1, \ldots, s_k \twoheadrightarrow t_k$$

- Reduction $\to_{\mathcal{R}}$ is defined as follows:

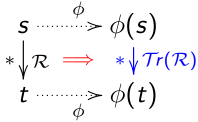
$$C[\ell\sigma] \to_{\mathcal{R}} C[r\sigma] \;\text{ iff }\; \left(\begin{array}{c} \exists \ell \to r \Leftarrow s_1 \twoheadrightarrow t_1, \ldots, s_k \twoheadrightarrow t_k \in \mathcal{R}. \\ s_1\sigma \to_{\mathcal{R}}^* t_1\sigma \;\wedge\; \cdots \;\wedge\; s_k\sigma \to_{\mathcal{R}}^* t_k\sigma \end{array} \right)$$

- $\mathcal{R}$ is 1-CTRS if $\forall \ell \to r \Leftarrow c \in \mathcal{R}.\ \mathcal{V}ar(\ell) \supseteq \mathcal{V}ar(r) \cup \mathcal{V}ar(c)$
- $\mathcal{R}$ is 3-CTRS if $\forall \ell \to r \Leftarrow c \in \mathcal{R}.\ \mathcal{V}ar(\ell) \cup \mathcal{V}ar(c) \supseteq \mathcal{V}ar(r)$
- $\mathcal{R}$ is deterministic (DCTRS) if
  $\forall \ell \to r \Leftarrow s_1 \twoheadrightarrow t_1, \ldots, s_k \twoheadrightarrow t_k.\ \forall i.\ \mathcal{V}ar(s_i) \subseteq \mathcal{V}ar(\ell, t_1, \ldots, t_{i-1})$
  - Evaluate conditions left to right

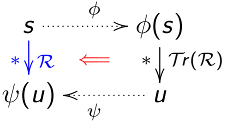
Analysis of Properties for CTRSs

- Much more complicated than that of TRSs
- Transformational approach
  - Let $\mathcal{T}r$ be a transformation from CTRSs $\mathcal{R}$ over $\mathcal{F}$ into TRSs over $\mathcal{G}$
    - ★ Signatures may be modified via mapping ϕ and its partial inverse ψ
    - ★ $\phi : T(\mathcal{F},\mathcal{V}) \mapsto T(\mathcal{G},\mathcal{V})$
    - ★ $\psi : T(\mathcal{G},\mathcal{V}) \mapsto T(\mathcal{F},\mathcal{V})$ s.t. $\forall t \in T(\mathcal{F},\mathcal{V}).\ \psi(\phi(t)) = t$
  - Required to be complete, but not sound for all CTRSs

$\mathcal{T}r$ (and $\mathcal{T}r(\mathcal{R})$) is complete for $\mathcal{R}$



$\mathcal{T}r$ (and $\mathcal{T}r(\mathcal{R})$) is sound for $\mathcal{R}$



  - Analyze $\mathcal{T}r(\mathcal{R})$ instead of $\mathcal{R}$ using techniques for TRSs
- Two well investigated transformations
  - unravelings $\mathbb{U}$ [Marchiori, 96] [...]
  - transformations based on Viry's one [Viry, 99] [...]

Example of Normal 1-CTRS

$$\mathcal{R}_1 = \left\{ \begin{array}{rcl} \mathsf{e}(0) & \to & \mathsf{true} \\ \mathsf{e}(\mathsf{s}(x)) & \to & \mathsf{true} \Leftarrow \mathsf{e}(x) \twoheadrightarrow \mathsf{false} \\ \mathsf{e}(\mathsf{s}(x)) & \to & \mathsf{false} \Leftarrow \mathsf{e}(x) \twoheadrightarrow \mathsf{true} \end{array} \right\}$$

$$\mathbb{U}(\mathcal{R}_1) = \left\{ \begin{array}{rclrcl} \mathsf{e}(0) & \to & \mathsf{true} & & & \\ \mathsf{e}(\mathsf{s}(x)) & \to & U_1(\mathsf{e}(x), x) & U_1(\mathsf{false}, x) & \to & \mathsf{true} \\ \mathsf{e}(\mathsf{s}(x)) & \to & U_2(\mathsf{e}(x), x) & U_2(\mathsf{true}, x) & \to & \mathsf{false} \end{array} \right\}$$

$$\mathbb{SR}(\mathcal{R}_1) = \left\{ \begin{array}{rcl} \overline{\mathsf{e}}(0, z_1, z_2) & \to & \langle \mathsf{true} \rangle \\ \overline{\mathsf{e}}(\mathsf{s}(x), \bot, z_2) & \to & \overline{\mathsf{e}}(\mathsf{s}(x), \langle \overline{\mathsf{e}}(x, \bot, \bot) \rangle, z_2) \\ \overline{\mathsf{e}}(\mathsf{s}(x), \langle \mathsf{false} \rangle, z_2) & \to & \langle \mathsf{true} \rangle \\ \overline{\mathsf{e}}(\mathsf{s}(x), z_1, \bot) & \to & \overline{\mathsf{e}}(\mathsf{s}(x), z_1, \langle \overline{\mathsf{e}}(x, \bot, \bot) \rangle) \\ \overline{\mathsf{e}}(\mathsf{s}(x), z_1, \langle \mathsf{true} \rangle) & \to & \langle \mathsf{false} \rangle \\ & & \\ \langle \langle x \rangle \rangle & \to & \langle x \rangle \\ \mathsf{s}(\langle x \rangle) & \to & \langle \mathsf{s}(x) \rangle \\ \overline{\mathsf{e}}(\langle x \rangle, z_1, z_2) & \to & \langle \overline{\mathsf{e}}(x, \bot, \bot) \rangle \end{array} \right\}$$

Example of 3-DCTRS

$$\mathcal{R} = \left\{ \begin{array}{rl} \mathsf{split}(x, \mathsf{nil}) & \to \mathsf{pair}(\mathsf{nil}, \mathsf{nil}) \\ \mathsf{split}(x, \mathsf{cons}(y, ys)) & \to \mathsf{pair}(xs, \mathsf{cons}(y, zs)) \Leftarrow \mathsf{split}(x, ys) \twoheadrightarrow \mathsf{pair}(xs, zs),\ x \leq y \twoheadrightarrow \mathsf{true} \\ \mathsf{split}(x, \mathsf{cons}(y, ys)) & \to \mathsf{pair}(\mathsf{cons}(y, xs), zs) \Leftarrow \mathsf{split}(x, ys) \twoheadrightarrow \mathsf{pair}(xs, zs),\ x \leq y \twoheadrightarrow \mathsf{false} \\ \mathsf{qsort}(\mathsf{nil}) & \to \mathsf{nil} \\ \mathsf{qsort}(\mathsf{cons}(x, xs)) & \to \mathsf{qsort}(ys) +\!\!+ \mathsf{cons}(x, \mathsf{qsort}(zs)) \Leftarrow \mathsf{split}(x, xs) \twoheadrightarrow \mathsf{pair}(ys, zs) \\ & \vdots \end{array} \right\}$$

$$\mathbb{SR}(\mathcal{R}) = \left\{ \begin{array}{rl} \overline{\mathsf{split}}(x, \mathsf{nil}, z_1, z_2) & \to \langle \mathsf{pair}(\mathsf{nil}, \mathsf{nil}) \rangle \\ \overline{\mathsf{split}}(x, \mathsf{cons}(y, ys), \bot, z_2) & \to \overline{\mathsf{split}}(x, \mathsf{cons}(y, ys), [\langle \overline{\mathsf{split}}(x, ys, \bot, \bot) \rangle]_1, z_2) \\ \overline{\mathsf{split}}(x, \mathsf{cons}(y, ys), [\langle \mathsf{pair}(xs, zs) \rangle]_1, z_2) & \to \overline{\mathsf{split}}(x, \mathsf{cons}(y, ys), [\langle x \leq y \rangle, xs, zs]_2, z_2) \\ \overline{\mathsf{split}}(x, \mathsf{cons}(y, ys), [\langle \mathsf{true} \rangle, xs, zs]_2, z_2) & \to \langle \mathsf{pair}(xs, \mathsf{cons}(y, zs)) \rangle \\ \overline{\mathsf{split}}(x, \mathsf{cons}(y, ys), z_1, \bot) & \to \overline{\mathsf{split}}(x, \mathsf{cons}(y, ys), z_1, [\langle \overline{\mathsf{split}}(x, ys, \bot, \bot) \rangle]_3) \\ \overline{\mathsf{split}}(x, \mathsf{cons}(y, ys), z_1, [\langle \mathsf{pair}(xs, zs) \rangle]_3) & \to \overline{\mathsf{split}}(x, \mathsf{cons}(y, ys), z_1, [\langle x \leq y \rangle, xs, zs]_4) \\ \overline{\mathsf{split}}(x, \mathsf{cons}(y, ys), z_1, [\langle \mathsf{false} \rangle, xs, zs]_4) & \to \langle \mathsf{pair}(\mathsf{cons}(y, xs), zs) \rangle \\ \overline{\mathsf{qsort}}(\mathsf{nil}, z_1) & \to \langle \mathsf{nil} \rangle \\ \overline{\mathsf{qsort}}(\mathsf{cons}(x, xs), \bot) & \to \overline{\mathsf{qsort}}(\mathsf{cons}(x, xs), [\langle \overline{\mathsf{split}}(x, xs, \bot, \bot) \rangle]_5) \\ \overline{\mathsf{qsort}}(\mathsf{cons}(x, xs), [\langle \mathsf{pair}(ys, zs) \rangle]_5) & \to \langle \overline{\mathsf{qsort}}(ys, \bot) +\!\!+ \mathsf{cons}(x, \overline{\mathsf{qsort}}(zs, \bot)) \rangle \\ & \vdots \end{array} \right\}$$

Properties of CTRSs

- Term t is strongly irreducible if any normalized instance $t\sigma$ is in normal form of $\mathcal{R}$
- $\mathcal{R}$ is strongly deterministic if $\forall \ell \to r \Leftarrow s_1 \twoheadrightarrow t_1, \ldots, s_k \twoheadrightarrow t_k \in \mathcal{R}$, $t_1, \ldots, t_k$ are strongly irreducible
- $\mathcal{R}$ is syntactically deterministic if $\forall \ell \to r \Leftarrow s_1 \twoheadrightarrow t_1, \ldots, s_k \twoheadrightarrow t_k \in \mathcal{R}$, each t_i is a constructor term or ground strongly-irreducible
- $\mathcal{R}$ is normal if $\forall \ell \to r \Leftarrow s_1 \twoheadrightarrow t_1, \ldots, s_k \twoheadrightarrow t_k \in \mathcal{R}$, $t_1, \ldots, t_k$ are ground n.f. of $\mathcal{R}_u = \{\ell \to r \mid \ell \to r \Leftarrow c \in \mathcal{R}\}$
- $\mathcal{R}$ is an SDCTRS if $\mathcal{R}$ is strongly or syntactically deterministic
  - Normal 1-CTRSs are 3-SDCTRSs
- $\ell \to r \Leftarrow s_1 \twoheadrightarrow t_1, \ldots, s_k \twoheadrightarrow t_k$ is weakly left-linear (WLL) if $\forall x \in \mathcal{V}ar(r, s_1, \ldots, s_k).\ |\ell, t_1, \ldots, t_k|_x \leq 1$
  - No variables with non-linear occurrence in patterns $\ell, t_1, \ldots, t_k$ is not used in computation starting with any of $r, s_1, \ldots, s_k$

Unravelings $\mathbb{U}$

- Proposed for normal 1-CTRSs and 3-DCTRSs [Marchiori, 96]
- Modified for 3-DCTRSs [Ohlebusch, 01]
- $\mathcal{R}$ is ultra-P ($\mathbb{U}$-P) if $\mathbb{U}(\mathcal{R})$ is P [Marchiori, 96]
  - $\mathcal{R}$ is $\mathbb{U}$-(W)LL if $\mathbb{U}(R)$ is (W)LL
- Simple and thus many soundness results
  - $\mathbb{U}$-LL [Marchiori, 96], $\mathbb{U}$-LL [Nishida et al, 11], WLL [Gmeiner et al, 10, 12], etc
- Preserve non-operational-termination
  - If $\mathbb{U}(\mathcal{R})$ terminates, then $\mathcal{R}$ operationally (OP) terminates [Lucas et al, 05]
- Not always preserve confluence
- Improved for preserving confluence as much as possible [Gmeiner et al, 13]
  - If $\mathbb{U}(\mathcal{R})$ is sound for $\mathcal{R}$ and confluent, then $\mathcal{R}$ is confluent

Transformation $\mathbb{SR}$ [Şerbănuţă & Roşu, 06]

- The latest one of Viry's transformation
- Defined for normal 1-CTRSs and then for $\mathbb{U}$-LL 3-DCTRSs
- Preserve confluence and (non-)OP termination
  - $\mathbb{SR}$ is sound for $\mathbb{U}$-LL 3-SDCTRSs
  - If $\mathcal{R}$ is confluent, then $\mathbb{SR}(\mathcal{R})$ is confluent on reachable terms
    - ★ t is reachable if $\exists s \in T(\mathcal{F}, \mathcal{V}).\ \phi(s) \to^*_{\mathbb{SR}(\mathcal{R})} t$,
- Complicated but very efficient rewriting engine via $\mathbb{SR}(\mathcal{R})$
- Preserve non-OP termination and non-confluence
  - If $\mathbb{SR}$ is sound for $\mathcal{R}$ and confluent, then $\mathcal{R}$ is confluent [Nishida et al, 14]
  - $\mathbb{SR}$ is sound for WLL normal 1-CTRSs [Nishida et al, 14]
  - $\mathbb{SR}$ is defined for $\mathbb{U}$-WLL DCTRSs w/o any change [Nakayama et al, 16]
  - $\mathbb{SR}$ is sound for WLL and $\mathbb{U}$-WLL DCTRSs [Nakayama et al, 16]
    - ★ Not need to be SDCTRSs

Further Work for $\mathbb{U}$-WLL SDCTRSs

- Does $\mathbb{SR}$ preserve confluence of WLL and $\mathbb{U}$-WLL DCTRSs?
  - No
  - We show a counterexample
- Does $\mathbb{SR}$ preserve non-confluence of 3-DCTRSs as for normal 1-CTRSs?
  - Yes
  - We confirm that the following holds for $\mathbb{U}$-WLL SDCTRSs:
    - ⋆ If $\mathbb{SR}(\mathcal{R})$ is sound for $\mathcal{R}$ and confluent, then $\mathcal{R}$ is confluent
- Does $\mathbb{SR}$ preserve confluence of WLL and $\mathbb{U}$-WLL SDCTRSs?
  - Open problem

Counterexample to the Conjecture

Conjecture

If a WLL and $\mathbb{U}$-WLL DCTRS $\mathcal{R}$ is confluent, then $\mathbb{SR}(\mathcal{R})$ is confluent

- A counterexample exists

$$\mathcal{R} = \left\{ \begin{array}{rl} \mathsf{a} \to \mathsf{a} & \rho_1 : \quad \mathsf{f}(x) \to \mathsf{f}(\mathsf{a}) \Leftarrow x \twoheadrightarrow \mathsf{a} \\ \mathsf{a} \to \mathsf{b} & \rho_2 : \quad \mathsf{f}(x) \to \mathsf{b} \Leftarrow x \twoheadrightarrow \mathsf{b} \\ \mathsf{g}(\mathsf{g}(x,x),x) \to \mathsf{b} & \end{array} \right\}$$

$$\mathbb{SR}(\mathcal{R}) = \left\{ \begin{array}{rr} \overline{\mathsf{a}} \to \langle \overline{\mathsf{a}} \rangle & \overline{\mathsf{a}} \to \langle \overline{\mathsf{b}} \rangle \\ \overline{\mathsf{g}}(\overline{\mathsf{g}}(x,x),x) \to \langle \overline{\mathsf{b}} \rangle & \\ \overline{\mathsf{f}}(x,\bot,z_2) \to \overline{\mathsf{f}}(x,[\langle x \rangle]_1^{\rho_1},z_2) & \overline{\mathsf{f}}(x,z_1,\bot) \to \overline{\mathsf{f}}(x,z_1,[\langle x \rangle]_1^{\rho_2}) \\ \overline{\mathsf{f}}(x,[\langle \overline{\mathsf{a}} \rangle]_1^{\rho_1},z_2) \to \langle \overline{\mathsf{f}}(\overline{\mathsf{a}},\bot,\bot) \rangle & \overline{\mathsf{f}}(x,z_1,[\langle \overline{\mathsf{b}} \rangle]_1^{\rho_2}) \to \langle \overline{\mathsf{b}} \rangle \\ \langle\langle x \rangle\rangle \to \langle x \rangle & \overline{\mathsf{f}}(\langle x \rangle,z_1,z_2) \to \langle \overline{\mathsf{f}}(x,\bot,\bot) \rangle \\ \overline{\mathsf{g}}(\langle x \rangle,y) \to \langle \overline{\mathsf{g}}(x,y) \rangle & \overline{\mathsf{g}}(x,\langle y \rangle) \to \langle \overline{\mathsf{g}}(x,y) \rangle \end{array} \right\}$$

  - $\mathcal{R}$ is a WLL and $\mathbb{U}$-WLL DCTRS and thus $\mathbb{SR}$ is sound for $\mathcal{R}$
  - $\mathcal{R}$ is confluent but $\mathbb{SR}(\mathcal{R})$ is not confluent

Preserving Non-Confluence

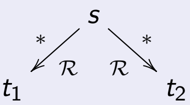
Theorem (normal 1-CTRSs [Nishida et al, 14] & $\mathbb{U}$-WLL DCTRSs)

If $\mathbb{SR}(\mathcal{R})$ is sound for $\mathcal{R}$ and confluent, then $\mathcal{R}$ is confluent

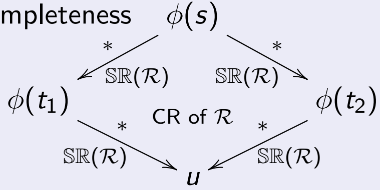
- Hold for $\mathbb{U}$-WLL DCTRSs

Proof.

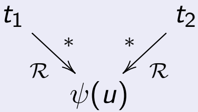
- Suppose that



By completeness



- By soundness



□

- The proof relies on completeness and soundness only

Experimental Results via CO3

- 51 normal 1-CTRSs and 54 3-DCTRSs from Cops
- The numbers of **YES** returned

	$\mathbb{U} + \mathbb{SR}$ for normal 1	$\mathbb{U}$ w/o opt. for 3-D	$\mathbb{U}$ w opt. for 3-D	$\mathbb{SR}$ **for 3-D**
normal 1	21	21	22	21
3-D	–	1	17	0

- Power of CO3 for proving termination is too weak
  - Any $\mathbb{SR}$-transformed TRS is not orthogonal
  - The only possible way is to prove termination and joinability of CPs

Conclusion

- Showed a counterexample to the conjecture for WLL and $\mathbb{U}$-WLL DCTRSs even on reachable terms
- Confirmed that the confluence criterion works for $\mathbb{U}$-WLL DCTRSs
  - For a WLL and $\mathbb{U}$-WLL DCTRS $\mathcal{R}$, if $\mathbb{SR}(\mathcal{R})$ is confluent, then $\mathcal{R}$ is confluent
- Future work:
  - Experiments using other confluence provers
  - Prove the open problem

Open Problem

If a WLL and $\mathbb{U}$-WLL SDCTRS $\mathcal{R}$ is confluent, then $\mathbb{SR}(\mathcal{R})$ is confluent on reachable terms