An Algebraic Approach to Confluence and Completion

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  - interpret some algebraic constructions as completion algorithms.
- For that, we consider functional methods. These methods use reduction operators.

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Definition. An endomorphism T of $\mathbb{K}G$ is a reduction operator relative to $(G, <)$ if

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- $\forall g \in G$, we have $T(g) = g$ or $T(g) = \sum \lambda_i g_i$ with $g_i < g$.

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Notations.

- ▷ $\mathbf{RO}\,(G, <) := \{\text{reduction operators relative to } (G, <)\}$,
- ▷ $\forall T \in \mathbf{RO}\,(G, <)$,
  - ▷ $\mathrm{Red}\,(T) := \{g \in G,\ \ T(g) = g\}$,
  - ▷ $\mathrm{Nred}\,(T) := G \setminus \mathrm{Red}\,(T)$.

Lattice Structure

Theorem [C 2016]. *The map*

$$\ker\colon \mathbf{RO}(G,<) \longrightarrow \{\text{subspaces of } \mathbb{K}G\},$$
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- $(\mathbf{RO}\,(G,<)\,, \preceq, \wedge, \vee)$ is a lattice where
  - $T_1 \preceq T_2$ if $\ker\,(T_2) \ \subset \ \ker\,(T_1)$,
  - $T_1 \wedge T_2 := \theta\,(\ker(T_1) + \ker(T_2))$,
  - $T_1 \vee T_2 := \theta\,(\ker(T_1) \cap \ker(T_2))$.

Confluence

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$$T_1 \preceq T_2 \Longrightarrow \mathrm{Red}(T_1) \subset \mathrm{Red}(T_2).$$

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Definition. F is said to be confluent if Obs^F is the empty set.

The Abstract Rewriting System Associated with F

- Let $\left(\mathbb{K}G, \underset{F}{\longrightarrow}\right)$ defined by $v \underset{F}{\longrightarrow} T(v)$ for $T \in F$ such that $v \notin \mathbb{K}\mathrm{Red}\,(T)$.

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- **Fact:** $\wedge F(v)$ is the smallest element of $[v]$.
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Proposition. *F is confluent if and only if it is so for $\underset{F}{\longrightarrow}$.*

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Proposition. *Let $C \in \mathbf{RO}(G,<)$ such that $(\wedge F) \wedge C = \wedge F$. Then,*

$$C \text{ is a complement of } F \iff F \cup \{C\} \text{ is a completion of } F.$$

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Theorem [C 2016]. *The F-complement is a minimal complement of F.*

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- Thank you for listening.