Exercise 1 : Fixpoints
Consider the following syntax for types:

\[ A, B ::= a | A \lor B | A \land B | A \rightarrow B | \top | \bot | \text{Fix}_a.A \]

where the last three constructs are the constants true, false, and a fixpoint construct that binds \( a \) in \( A \). Types are quotiented by \( a \)-renaming, and by the equation \( \text{Fix}_a.A = \{ \text{Fix}_a.A \setminus \{a\} \} \).

Using that syntax for types, we work in Curien-Herbelin-Wadler calculus; we just give ourselves the constant term \( \star : \top \) and the constant continuation \( \text{top} : \bot \).

1. Let \( A \) be an arbitrary type. Give a closed term of type \( \top \lor (A \land a) \).
2. Give a closed term of type \( \text{Fix}_a.(\top \lor (A \land a)) \).
3. Given a term \( t \) of type \( A \), give a term of type \( \text{Fix}_a.(\top \lor (A \land a)) \) that has \( t \) as one of its sub-terms.
4. Assuming \( A \) is non-empty, describe infinitely many terms of type \( \text{Fix}_a.(\top \lor (A \land a)) \).
5. Let \( A\text{List} \) be an abbreviation for \( \text{Fix}_a.(\top \lor (A \land a)) \). Give a term \( \text{El} : A\text{List} \) representing the empty list and a construct \( \text{Cons} \) such that \( \text{Cons}(t, l) : A\text{List} \) represents the list of head \( t \) and of tail \( l \).
6. Assume that you now have a mechanism for raising exceptions: a term constant \( \text{Exception} : \bot \). Give a typing derivation for \( \langle \text{Exception} \bullet \text{top} \rangle \).
7. Consider the usual reduction system for Curien-Herbelin-Wadler calculus.
   Write a term \( \text{head} \) that returns the head of a non-empty list:
   i.e. such that \( \langle \text{head} \bullet \text{Cons}(t, l) :: e \rangle \rightarrow^* \langle l \bullet e \rangle \)
   and that raises an exception when applied to the empty list \( \langle \text{head} \bullet \text{El} :: e \rangle \rightarrow^* \langle \text{Exception} \bullet \text{top} \rangle \)
   Give a typing for \( \text{head} \).
8. Similarly, write a term \( \text{tail} \) that returns the head of a non-empty list:
   i.e. such that \( \langle \text{tail} \bullet \text{Cons}(t, l) :: e \rangle \rightarrow^* \langle l \bullet e \rangle \)
   and that raises an exception when applied to the empty list \( \langle \text{tail} \bullet \text{El} :: e \rangle \rightarrow^* \langle \text{Exception} \bullet \text{top} \rangle \)
   Give a typing for \( \text{tail} \).
9. Let \( c \) be a command and \( y \) a variable not free in \( c \).
   What are the CBV-reducts and normal forms of \( \langle \text{head} \bullet \text{El} :: \mu y.c \rangle \)?
   What are the CBN-reducts and normal forms of \( \langle \text{head} \bullet \text{El} :: \mu y.c \rangle \)?