Part II

Barbanera and Berardi's proof of strong normalisation for 2nd-order classical logic

Barbanera and Berardi's calculus is a 1-sided version of Curien-Herbelin-Wadler's: terms and continuations are merged into 1 syntax (this gives half as many cases to treat in your proofs).

$$
\begin{array}{llcl}
\textbf{Types} & A, B, C, \ldots & ::= & \alpha \mid \alpha^\perp \mid A \vee B \mid A \wedge B \mid \exists \alpha.A \mid \forall \alpha.A \\
\textbf{Terms} & t, u, v, \ldots & ::= & x \mid \mu x.p \mid \mathsf{inj}_1(t) \mid \mathsf{inj}_2(t) \mid \langle t, u\rangle \mid \langle \_, t\rangle \mid \Lambda\_.t \\
\textbf{Commands} & p & ::= & \langle t \bullet u \rangle
\end{array}
$$

Negation:

$$
\begin{array}{lcl}
(\alpha^\perp)^\perp & := & \alpha \\
(A \wedge B)^\perp & := & A^\perp \vee B^\perp \\
(A \vee B)^\perp & := & A^\perp \wedge B^\perp \\
(\forall \alpha.A)^\perp & := & \exists \alpha.A^\perp \\
(\exists \alpha.A)^\perp & := & \forall \alpha.A^\perp
\end{array}
$$

Typing rules:

$$
\frac{}{\Gamma \vdash x : A}(x : A) \in \Gamma \qquad \frac{\Gamma, x : A \vdash p}{\Gamma \vdash \mu x.p : A^\perp} \qquad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : A^\perp}{\Gamma \vdash \langle t \bullet u \rangle}
$$

$$
\frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B}{\Gamma \vdash \langle t, u \rangle : A \wedge B} \qquad \frac{\Gamma \vdash t : A_i}{\Gamma \vdash \mathsf{inj}_i(t) : A_1 \vee A_2}
$$

$$
\frac{\Gamma \vdash t : B}{\Gamma \vdash \Lambda\_.t : \forall \alpha.B}\alpha \notin \mathsf{FV}(\Gamma) \qquad \frac{\Gamma \vdash u : \{{}^{B}\!/\!{}_{\alpha}\}A}{\Gamma \vdash \langle \_, u \rangle : \exists \alpha.A}
$$

The following reduction rules apply anywhere in terms and commands:

$$
\begin{array}{lcllcl}
\langle \mu x.p \bullet t \rangle & \longrightarrow & \{{}^{t}\!/\!{}_{x}\}p & \langle t \bullet \mu x.p \rangle & \longrightarrow & \{{}^{t}\!/\!{}_{x}\}p \\
\langle \langle t_1, t_2 \rangle \bullet \mathsf{inj}_i(t) \rangle & \longrightarrow & \langle t_i \bullet t \rangle & \langle \mathsf{inj}_i(t) \bullet \langle t_1, t_2 \rangle \rangle & \longrightarrow & \langle t \bullet t_i \rangle \\
\langle \Lambda\_.t \bullet \langle \_, u \rangle \rangle & \longrightarrow & \langle t \bullet u \rangle & \langle \langle \_, u \rangle \bullet \Lambda\_.t \rangle & \longrightarrow & \langle u \bullet t \rangle
\end{array}
$$

Let Var denote the set of term variables, and SN (resp. SN^c) denote the set of strongly normalising terms (resp. commands) for the reduction relation induced by the above rules.
Notice that the calculus does not satisfy confluence, with the obvious critical pair:

$$
\begin{array}{ccc}
& \langle \mu x.p_1 \bullet \mu y.p_2 \rangle & \\
\swarrow & & \searrow \\
\{{}^{\mu y.p_2}\!/\!{}_{x}\}p_1 & & \{{}^{\mu x.p_1}\!/\!{}_{y}\}p_2
\end{array}
$$

Exercise 1 : Orthogonality and saturation

We start with a few definitions:

- $t \perp u$ if $\langle t \bullet u \rangle \in \mathsf{SN}^c$
- A pair $(\mathcal{U}, \mathcal{V})$ of sets of terms is
  - *orthogonal* if $\forall t \in \mathcal{U}, \forall u \in \mathcal{V}, t \perp u$
  - *saturated* if the following two conditions hold
    1. $\mathsf{Var} \subseteq \mathcal{U}$ and $\mathsf{Var} \subseteq \mathcal{V}$
    2. $\{\mu x.\langle t \bullet u \rangle \mid \forall v \in \mathcal{V}, \{{}^{v}\!/\!{}_{x}\}t \perp \{{}^{v}\!/\!{}_{x}\}u\} \subseteq \mathcal{U}$ and
       $\{\mu x.\langle t \bullet u \rangle \mid \forall v \in \mathcal{U}, \{{}^{v}\!/\!{}_{x}\}t \perp \{{}^{v}\!/\!{}_{x}\}u\} \subseteq \mathcal{V}$.

1. [*] Briefly justify (no full proof required) why
   - $(\mathcal{U}, \mathsf{Var})$ is orthogonal if and only if $\mathcal{U} \subseteq \mathsf{SN}$.

     Correction : Assume $(\mathcal{U}, \mathsf{Var})$ is orthogonal; for all $u \in \mathcal{U}$, take $x \in \mathsf{Var}$; since $u \perp x$ we have $u \in \mathsf{SN}$. Conversely for all $u \in \mathcal{U} \subseteq \mathsf{SN}$ and $x \in \mathsf{Var}$, $\langle u \bullet x \rangle \in \mathsf{SN}^c$.
   - If $\{{}^{v}\!/\!{}_{x}\}\langle t \bullet u \rangle \in \mathsf{SN}^c$ then we have $\langle t \bullet u \rangle \in \mathsf{SN}^c$.

     Correction : As reduction is stable under substitution, an infinite reduction sequence starting from $\langle t \bullet u \rangle$ would provide one starting from $\{{}^{v}\!/\!{}_{x}\}\langle t \bullet u \rangle$.
   - If $\{{}^{v}\!/\!{}_{x}\}\langle t \bullet u \rangle \in \mathsf{SN}^c$ and $v \in \mathsf{SN}$ is not of the form $\mu x.p$,
     then we have $\langle v \bullet \mu x.\langle t \bullet u \rangle \rangle \in \mathsf{SN}^c$ and $\langle \mu x.\langle t \bullet u \rangle \bullet v \rangle \in \mathsf{SN}^c$.

     Correction : If $\{{}^{v}\!/\!{}_{x}\}\langle t \bullet u \rangle \in \mathsf{SN}^c$ then from the previous point $\langle t \bullet u \rangle \in \mathsf{SN}^c$ and $\mu x.\langle t \bullet u \rangle \in \mathsf{SN}$, so as $v \in \mathsf{SN}$ as well, an infinite reduction sequence starting from $\langle v \bullet \mu x.\langle t \bullet u \rangle \rangle$ has to reduce, at some point, the top-level redex, and the only possible way is to $\left\{{}^{v'}\!/\!_x\right\}\langle t' \bullet u'\rangle$ for some reduced forms t', u', v' of t, u, v; this would provide an infinite reduction sequence from $\{{}^{v}\!/\!_x\}\langle t \bullet u\rangle$.

A similar reasoning proves $\langle \mu x.\langle t \bullet u\rangle \bullet v\rangle \in \mathsf{SN}^c$, or alternatively we can use the symetry of the calculus: given the reduction rules, a command $\langle u \bullet v\rangle \in \mathsf{SN}^c$ if and only if$\langle v \bullet u\rangle \in \mathsf{SN}^c$.

2. Given an orthogonal pair $(\mathcal{U}, \mathcal{V})$ of non-empty sets, prove that $\mathcal{U} \subseteq \mathsf{SN}$ and $\mathcal{V} \subseteq \mathsf{SN}$.

   Correction : For all $u \in \mathcal{U}$, take v the non-empty set $\mathcal{V}$; since $u \perp v$ we have $u \in \mathsf{SN}$. Same argument for $\mathcal{V}$.

Exercise 2 : Saturated extensions of simple pairs

A set of terms is said to be *simple* if it is non-empty and it contains no term of the form $\mu x.t$.
In this exercise we want to build a "saturating" function *satur*, i.e. a function such that, for any orthogonal pair $(\mathcal{U}, \mathcal{V})$ of simple sets, $\mathsf{satur}(\mathcal{U}, \mathcal{V})$ is a saturated and orthogonal pair of sets $(\mathcal{U}', \mathcal{V}')$ that extends $(\mathcal{U}, \mathcal{V})$ (i.e. such that $\mathcal{U} \subseteq \mathcal{U}'$ and $\mathcal{V} \subseteq \mathcal{V}'$).

1. For every set $\mathcal{U}$ of terms, we define a function
$$\Phi_{\mathcal{U}}(\mathcal{W}) := \mathcal{U} \cup \mathsf{Var} \cup \{\mu x.\langle t \bullet u\rangle \mid \forall v \in \mathcal{W}, \{{}^{v}\!/\!_x\}t \perp \{{}^{v}\!/\!_x\}u\}$$
   Prove that $\Phi_{\mathcal{U}}$ is anti-monotonic (i.e. if $\mathcal{W} \subseteq \mathcal{W}'$ then $\Phi_{\mathcal{U}}(\mathcal{W}) \supseteq \Phi_{\mathcal{U}}(\mathcal{W}')$).

   Correction : If $\mathcal{W} \subseteq \mathcal{W}'$ then for any t, u, $\forall v \in \mathcal{W}', \{{}^{v}\!/\!_x\}t \perp \{{}^{v}\!/\!_x\}u$ implies $\forall v \in \mathcal{W}, \{{}^{v}\!/\!_x\}t \perp \{{}^{v}\!/\!_x\}u$, so $\Phi_{\mathcal{U}}(\mathcal{W}') \subseteq \Phi_{\mathcal{U}}(\mathcal{W})$.

2. Given two sets of terms $\mathcal{U}$ and $\mathcal{V}$, prove that $\Phi_{\mathcal{U}} \circ \Phi_{\mathcal{V}}$ admits a fixed point (a set $\mathcal{U}'$ such that $\Phi_{\mathcal{U}}(\Phi_{\mathcal{V}}(\mathcal{U}')) = \mathcal{U}'$).

   Correction : From the previous question, $\Phi_{\mathcal{U}} \circ \Phi_{\mathcal{V}}$ is a monotonic set transformation, so it admits the fixpoint $\bigcup_{n \in \mathbb{N}} (\Phi_{\mathcal{U}} \circ \Phi_{\mathcal{V}})^n(\emptyset)$.

3. Let $\mathcal{U}'$ be a fixed point of $\Phi_{\mathcal{U}} \circ \Phi_{\mathcal{V}}$, and let $\mathcal{V}' := \Phi_{\mathcal{V}}(\mathcal{U}')$. Prove the following:
$$\begin{aligned} \mathcal{U}' &= \mathcal{U} \cup \mathsf{Var} \cup \{\mu x.\langle t \bullet u\rangle \mid \forall v \in \mathcal{V}', \{{}^{v}\!/\!_x\}t \perp \{{}^{v}\!/\!_x\}u\} \\ \mathcal{V}' &= \mathcal{V} \cup \mathsf{Var} \cup \{\mu x.\langle t \bullet u\rangle \mid \forall v \in \mathcal{U}', \{{}^{v}\!/\!_x\}t \perp \{{}^{v}\!/\!_x\}u\} \end{aligned}$$

   Correction : Just by unfolding the definitions: the first equality is the unfolding of the fixpoint equality $\mathcal{U}' = \Phi_{\mathcal{U}} \circ \Phi_{\mathcal{V}}(\mathcal{U}')$; then second one is the unfolding of $\mathcal{V}' := \Phi_{\mathcal{V}}(\mathcal{U}')$.

4. Prove that the pair $(\mathcal{U}', \mathcal{V}')$ is saturated and extends $(\mathcal{U}, \mathcal{V})$.

   Correction : This can be read directly on the above equations.

5. [*] Assume that $(\mathcal{U}, \mathcal{V})$ is an orthogonal pair of simple sets; prove that the pair $(\mathcal{U}', \mathcal{V}')$ is orthogonal.

   Correction : First, notice that as $\mathcal{U}$ and $\mathcal{V}$ are assumed simple, they are in particular non-empty and, by Ex.1-Q.2, $\mathcal{U} \subseteq \mathsf{SN}$ and $\mathcal{V} \subseteq \mathsf{SN}$.

   Second, by induction on $n \in \mathbb{N}$, notice that $(\Phi_{\mathcal{U}} \circ \Phi_{\mathcal{V}})^n(\emptyset) \subseteq \mathsf{SN}$ and $\Phi_{\mathcal{V}}(\Phi_{\mathcal{U}} \circ \Phi_{\mathcal{V}})^n(\emptyset) \subseteq \mathsf{SN}$ (each induction step uses Ex.1-Q.1.2).

   Third, we conclude from this that $\mathcal{U}' \subseteq \mathsf{SN}$ and $\mathcal{V}' \subseteq \mathsf{SN}$.

   Now let $u \in \mathcal{U}'$ and $v \in \mathcal{V}'$. We show $u \perp v$ by case analysis on the different subsets composing $\mathcal{U}'$ and $\mathcal{V}'$:

$u \backslash v$	$\mathcal{V}$	Var	$\{\mu x.\langle v1 \bullet v2\rangle \mid \forall u \in \mathcal{U}', \{{}^{u}\!/\!_x\}v1 \perp \{{}^{u}\!/\!_x\}v2\}$
$\mathcal{U}$	$(\mathcal{U}, \mathcal{V})$ assumed orthogonal	1	2
Var	1	No reduction	2
$\{\mu x.\langle u1 \bullet u2\rangle \mid \forall v \in \mathcal{V}', \{{}^{v}\!/\!_x\}u1 \perp \{{}^{v}\!/\!_x\}u2\}$	3	3	4

   1: By Ex.1-Q.1.1, $(\mathcal{U}, \mathsf{Var})$ and $(\mathcal{V}, \mathsf{Var})$ are orthogonal pairs.

   2 and 3: By Ex.1-Q.1.3.

   4: This is the interesting case: $u = \mu x.\langle u1 \bullet u2\rangle$ and $v = \mu x.\langle v1 \bullet v2\rangle$.

   Assume there is an infinite reduction sequence from $\langle \mu x.\langle u1 \bullet u2\rangle \bullet \mu y.\langle v1 \bullet v2\rangle\rangle$. As both

$\mu x.\langle u1 \bullet u2\rangle \in \mathcal{U}' \subseteq \mathsf{SN}$ and $\mu y.\langle v1 \bullet v2\rangle \in \mathcal{V}' \subseteq \mathsf{SN}$, the reduction sequence must reduce the top-level redex at some point, to either $\left\{{}^{\mu y.\langle v1' \bullet v2'\rangle}\!/\!_{x}\right\}\langle u1' \bullet u2'\rangle$ or $\left\{{}^{\mu x.\langle u1' \bullet u2'\rangle}\!/\!_{y}\right\}\langle v1' \bullet v2'\rangle$ for some reduced forms $u1'$, $u2'$, $v1'$, $v2'$ of $u1$, $u2$, $v1$, $v2$.

We treat both cases: $\left\{{}^{\mu y.\langle v1' \bullet v2'\rangle}\!/\!_{x}\right\}\langle u1' \bullet u2'\rangle$ is a reduced form of $\{{}^{y}\!/\!_{x}\}\langle u1 \bullet u2\rangle$, which is in SN; $\left\{{}^{\mu x.\langle u1' \bullet u2'\rangle}\!/\!_{y}\right\}\langle v1' \bullet v2'\rangle$ is a reduced form of $\{{}^{y}\!/\!_{y}\}\langle v1 \bullet v2\rangle$, which is in SN.

We finally define $\mathsf{satur}(\mathcal{U}, \mathcal{V}) := (\mathcal{U}', \mathcal{V}')$.

Exercise 3 : Semantics

The general idea: a type A is interpreted

- first as an orthogonal pair of simple sets $\qquad [A]_\sigma = ([A]^+_\sigma, [A]^-_\sigma)$
- then as an orthogonal and saturated pair of sets $\qquad [\![A]\!]_\sigma = ([\![A]\!]^+_\sigma, [\![A]\!]^-_\sigma)$

where we write $\mathcal{P}^+$ (resp. $\mathcal{P}^-$) for the first (resp. second) component of a pair $\mathcal{P}$.

Let $\mathcal{H}$ be the set of all *orthogonal pairs of simple sets* (of terms).
A *valuation* σ is a mapping from type variables to $\mathcal{H}$.

1. Given some sets of terms $\mathcal{U}$ and $\mathcal{V}$, we define the following set constructions:

$$\begin{array}{lcllcl} \mathsf{inj}_1(\mathcal{U}) & := & \{\mathsf{inj}_1(t) \mid t \in \mathcal{U}\} & \langle \mathcal{U}, \mathcal{V}\rangle & := & \{\langle u, v\rangle \mid u \in \mathcal{U}, v \in \mathcal{V}\} \\ \mathsf{inj}_2(\mathcal{U}) & := & \{\mathsf{inj}_2(t) \mid t \in \mathcal{U}\} & & & \\ \langle \_, \mathcal{U}\rangle & := & \{\langle \_, u\rangle \mid u \in \mathcal{U}\} & \Lambda\_.\mathcal{U} & := & \{\Lambda\_.u \mid u \in \mathcal{U}\} \end{array}$$

   Prove that those sets are always simple if $\mathcal{U}$ and $\mathcal{V}$ are non-empty.

   Correction : None of these sets contains any term of the form $\mu x.c$, and none of them is empty if $\mathcal{U}$ and $\mathcal{V}$ are non-empty.

   We define the following semantics $[\_]_\_$ and $[\![\_]\!]_\_$:

$$\boxed{\begin{array}{lcllcl} [\alpha]^+_\sigma & := & \sigma(\alpha)^+ & [\alpha]^-_\sigma & := & \sigma(\alpha)^- \\ [A \vee B]^+_\sigma & := & \mathsf{inj}_1([\![A]\!]^+_\sigma) \cup \mathsf{inj}_2([\![B]\!]^+_\sigma) & [A \vee B]^-_\sigma & := & \langle [\![A]\!]^-_\sigma, [\![B]\!]^-_\sigma\rangle \\ [\exists\alpha.A]^+_\sigma & := & \langle \_, \bigcup_{h\in\mathcal{H}} [\![A]\!]^+_{\sigma,\alpha\mapsto h}\rangle & [\exists\alpha.A]^-_\sigma & := & \Lambda\_.\bigcap_{h\in\mathcal{H}} [\![A]\!]^-_{\sigma,\alpha\mapsto h} \\ ([A]^+_\sigma, [A]^-_\sigma) & := & ([A^\perp]^-_\sigma, [A^\perp]^+_\sigma) & \multicolumn{3}{l}{\text{if } A \text{ of the form } \alpha^\perp,\ A_1 \wedge A_2,\ \forall\alpha.A} \\ & & & & & \\ ([\![A]\!]^+_\sigma, [\![A]\!]^-_\sigma) & := & \mathsf{satur}([A]^+_\sigma, [A]^-_\sigma) & \multicolumn{3}{l}{\text{if } A \text{ of the form } \alpha,\ A_1 \vee A_2,\ \exists\alpha.A} \\ ([\![A]\!]^+_\sigma, [\![A]\!]^-_\sigma) & := & ([\![A^\perp]\!]^-_\sigma, [\![A^\perp]\!]^+_\sigma) & \multicolumn{3}{l}{\text{if } A \text{ of the form } \alpha^\perp,\ A_1 \wedge A_2,\ \forall\alpha.A} \end{array}}$$

2. Prove that if $[A]_\sigma = (\mathcal{U}, \mathcal{V})$ then $\left[A^\perp\right]_\sigma = (\mathcal{V}, \mathcal{U})$, and if $[\![A]\!]_\sigma = (\mathcal{U}, \mathcal{V})$ then $\left[\!\left[A^\perp\right]\!\right]_\sigma = (\mathcal{V}, \mathcal{U})$.

   Correction : If A of the form α, $A_1 \vee A_2$, $\exists\alpha.A$, the first point is by line 4 and the second point is by line 6. If A of the form $\alpha^\perp$, $A_1 \wedge A_2$, $\forall\alpha.A$, it is the same lines again, noticing the involutivity of negation: $A^{\perp\perp} = A$.

3. Prove that $[A]_{\sigma,\alpha\mapsto[B]_\sigma} = \left[\{{}^{B}\!/\!_{\alpha}\}A\right]_\sigma$ and $[\![A]\!]_{\sigma,\alpha\mapsto[B]_\sigma} = \left[\!\left[\{{}^{B}\!/\!_{\alpha}\}A\right]\!\right]_\sigma$.

   Correction : By induction on A.

4. [*] Prove, by induction on A, that $[A]_\sigma$ is an orthogonal pair of simple sets and $[\![A]\!]_\sigma$ is a saturated and orthogonal pair extending $[A]_\sigma$.

   Correction : For $A = \alpha$, the first point is by definition of $\mathcal{H}$.

   For $A = A_1 \vee A_2$ or $A = \exists\alpha.A'$, $[A]_\sigma$ is a pair of simple sets by Q.1.

   To prove that $[A_1 \vee A_2]_\sigma$ is orthogonal, let $u \in \mathsf{inj}_1([\![A_1]\!]^+_\sigma) \cup \mathsf{inj}_2([\![A_2]\!]^+_\sigma)$ and $v \in \langle [\![A_1]\!]^-_\sigma, [\![A_2]\!]^-_\sigma\rangle$. We have $u = \mathsf{inj}_i(u_0)$ with $u_0 \in [\![A_i]\!]^+_\sigma$ for either $i = 1$ or $i = 2$, while $v = \langle v_1, v_2\rangle$ with $v_1 \in [\![A_1]\!]^-_\sigma$ and $v_2 \in [\![A_2]\!]^-_\sigma$.

   Now the induction hypothesis gives that $[\![A_1]\!]_\sigma$ and $[\![A_2]\!]_\sigma$ are orthogonal pairs extending the pairs $[A_1]_\sigma$ and $[A_2]_\sigma$ of non-empty sets. So $[\![A_1]\!]^+_\sigma$, $[\![A_2]\!]^+_\sigma$, $[\![A_1]\!]^-_\sigma$, $[\![A_2]\!]^-_\sigma$ are themselves non-empty, and by Ex.1-Q.2 they are all included in SN. Hence, u and v are in SN. So an infinite reduction sequence starting from $\langle u \bullet v\rangle$ must reduce the top-level redex at some point, to $\langle u_0' \bullet v_i'\rangle$ for some reduced forms u_0', v_i' of u_0, v_i. This gives an infinite reduction sequence from $\langle u_0 \bullet v_i\rangle$ which contradicts the orthogonality of $[\![A_i]\!]_\sigma$.

To prove that $[\exists\alpha.A']_\sigma$ is orthogonal, let $u \in \langle\_, \bigcup_{h\in\mathcal{H}} [\![A]\!]^+_{\sigma,\alpha\mapsto h}\rangle$ and $v \in \Lambda\_.\bigcap_{h\in\mathcal{H}} [\![A]\!]^-_{\sigma,\alpha\mapsto h}$. We have $u = \langle\_, u_0\rangle$ with $u_0 \in [\![A']\!]^+_{\sigma,\alpha\mapsto h_0}$ for some $h_0 \in \mathcal{H}$, while $v = \Lambda\_.v_0$ with $v_0 \in [\![A']\!]^-_{\sigma,\alpha\mapsto h}$ for any $h \in \mathcal{H}$, in particular $h = h_0$.

Now the induction hypothesis gives that $[\![A']\!]_{\sigma,\alpha\mapsto h_0}$ is an orthogonal pair extending the pair $[A']_{\sigma,\alpha\mapsto h_0}$ of non-empty sets. So $[\![A']\!]^+_{\sigma,\alpha\mapsto h_0}$ and $[\![A']\!]^-_{\sigma,\alpha\mapsto h_0}$ are themselves non-empty, and by Ex.1-Q.2 they are each included in SN. Hence, u and v are in SN. So an infinite reduction sequence starting from $\langle u \bullet v\rangle$ must reduce the top-level redex at some point, to $\langle u_0' \bullet v_0'\rangle$ for some reduced forms u_0', v_0' of u_0, v_0. This gives an infinite reduction sequence from $\langle u_0 \bullet v_0\rangle$ which contradicts the orthogonality of $[\![A']\!]_{\sigma,\alpha\mapsto h_0}$.

For $A = \alpha^\perp$, $A = A_1 \wedge A_2$, or $A = \forall\alpha.A'$, $[A]_\sigma$ is an orthogonal pair of simple sets because it is $([A^\perp]^-_\sigma, [A^\perp]^+_\sigma)$.

For $A = \alpha$, $A = A_1 \vee A_2$ or $A = \exists\alpha.A'$, $[\![A]\!]_\sigma$ is a saturated and orthogonal pair extending $[A]_\sigma$ by Ex.2-Q.4 and Ex.2-Q.5.

For $A = \alpha^\perp$, $A = A_1 \wedge A_2$, $A = \forall\alpha.A'$, $[\![A]\!]_\sigma$ a saturated and orthogonal pair extending $[\![A]\!]_\sigma$ because it is $([\![A^\perp]\!]^-_\sigma, [\![A^\perp]\!]^+_\sigma)$.

Exercise 4 : Proof of Strong Normalisation

A *substitution* ρ is a mapping from term variables to terms. Applying a substitution ρ to a term t (in a capture-avoiding way) yields a term denoted $t\rho$. For each typing context Γ we define the set

$$[\![\Gamma]\!]_\sigma \;:=\; \{\rho \mid \forall(x\!:\!A) \in \Gamma, \rho(x) \in [\![A]\!]^+_\sigma\}$$

1. [*] Prove the *Adequacy Lemma*:
   If $\Gamma \vdash t\!:\!A$, then for all valuation σ and all substitutions $\rho \in [\![\Gamma]\!]_\sigma$ we have $t\rho \in [\![A]\!]^+_\sigma$.

   Correction : By induction on the typing tree, with the following statement for commands: If $\Gamma \vdash c$, then for all valuation σ and all substitutions $\rho \in [\![\Gamma]\!]_\sigma$ we have $c\rho \in \mathsf{SN}^c$.

   - $$\frac{}{\Gamma \vdash x\!:\!A}(x : A) \in \Gamma$$

     Let σ be a valuation and $\rho \in [\![\Gamma]\!]_\sigma$. We have $x\rho = \rho(x) \in [\![A]\!]^+_\sigma$ by assumption that $\rho \in [\![\Gamma]\!]_\sigma$.

   - $$\frac{\Gamma, x : A \vdash p}{\Gamma \vdash \mu x.p\!:\!A^\perp}$$

     Let σ be a valuation and $\rho \in [\![\Gamma]\!]_\sigma$. We need to show $(\mu x.p)\rho \in [\![A]\!]^+_\sigma$. Let us rewrite $(\mu x.p)\rho$ as $\mu x.(p\rho)$ (avoiding variable capture). Since $[\![A]\!]_\sigma$ is a saturated pair (Ex.3-Q.4), it suffices to show that for all $v \in [\![A]\!]^-_\sigma$, we have $\{{}^v\!/\!_x\}(p\rho) \in \mathsf{SN}^c$. Let $v \in [\![A]\!]^-_\sigma = [\![A^\perp]\!]^+_\sigma$ (Ex.3-Q.2). Notice that $\{{}^v\!/\!_x\}(p\rho) = p(\rho, x \mapsto v)$, and that $(\rho, x \mapsto v) \in [\![\Gamma, x\!:\!A^\perp]\!]_\sigma$. The induction hypothesis concludes want we want.

   - $$\frac{\Gamma \vdash t\!:\!A \qquad \Gamma \vdash u\!:\!A^\perp}{\Gamma \vdash \langle t \bullet u\rangle}$$

     Let σ be a valuation and $\rho \in [\![\Gamma]\!]_\sigma$. The induction hypothesis gives $t\rho \in [\![A]\!]^+_\sigma$ and $u\rho \in [\![A^\perp]\!]^+_\sigma = [\![A]\!]^-_\sigma$. Since $[\![A]\!]_\sigma$ is an orthogonal pair (Ex.3-Q.4), $(\langle t \bullet u\rangle)\rho = \langle t\rho \bullet u\rho\rangle \in \mathsf{SN}^c$.

   -

$$\frac{\Gamma \vdash t\!:\!A \qquad \Gamma \vdash u\!:\!B}{\Gamma \vdash \langle t,u\rangle\!:\!A \wedge B}$$

Let σ be a valuation and $\rho \in [\![\Gamma]\!]_\sigma$. The induction hypothesis gives $t\rho \in [\![A]\!]_\sigma^+$ and $u\rho \in [\![B]\!]_\sigma^+$. So $\langle t\rho, u\rho\rangle \in [A \wedge B]_\sigma^+$, and since $[\![A \wedge B]\!]_\sigma$ extends $[A \wedge B]_\sigma$ (Ex.3-Q.4), we have $(\langle t,u\rangle)\rho = \langle t\rho, u\rho\rangle \in [\![A \wedge B]\!]_\sigma^+$.

•

$$\frac{\Gamma \vdash t\!:\!A_i}{\Gamma \vdash \mathsf{inj}_i(t)\!:\!A_1 \vee A_2}$$

Let σ be a valuation and $\rho \in [\![\Gamma]\!]_\sigma$. The induction hypothesis gives $t\rho \in [\![A_i]\!]_\sigma^+$. So $\mathsf{inj}_i(t\rho) \in [A_1 \vee A_2]_\sigma^+$, and since $[\![A_1 \vee A_2]\!]_\sigma$ extends $[A_1 \vee A_2]_\sigma$ (Ex.3-Q.4), we have $(\mathsf{inj}_i(t))\rho = \mathsf{inj}_i(t\rho) \in [\![A_1 \vee A_2]\!]_\sigma^+$.

•

$$\frac{\Gamma \vdash t\!:\!B}{\Gamma \vdash \Lambda\_.t\!:\!\forall\alpha.B}\alpha \notin \mathsf{FV}(\Gamma)$$

Let σ be a valuation and $\rho \in [\![\Gamma]\!]_\sigma$. Since $\alpha \notin \mathsf{FV}(\Gamma)$, $[\![\Gamma]\!]_\sigma = [\![\Gamma]\!]_{\sigma,\alpha\mapsto h}$ for any $h \in \mathcal{H}$. So we can apply the induction hypothesis to obtain $t\rho \in [\![B]\!]_{\sigma,\alpha\mapsto h}^+$ for any $h \in \mathcal{H}$. So $\Lambda\_.t\rho \in [\forall\alpha.B]_\sigma^+$, and since $[\![\forall\alpha.B]\!]_\sigma$ extends $[\forall\alpha.B]_\sigma$ (Ex.3-Q.4), we have $(\Lambda\_.t)\rho = \Lambda\_.t\rho \in [\![\forall\alpha.B]\!]_\sigma^+$.

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$$\frac{\Gamma \vdash t\!:\!\left\{{}^{B}\!/\!_{\alpha}\right\}A}{\Gamma \vdash \langle \_, t\rangle\!:\!\exists\alpha.A}$$

Let σ be a valuation and $\rho \in [\![\Gamma]\!]_\sigma$. So we can apply the induction hypothesis to obtain $t\rho \in \left[\!\left[\left\{{}^{B}\!/\!_{\alpha}\right\}A\right]\!\right]_\sigma^+$, so by Ex.3-Q.3, $t\rho \in [\![A]\!]_{\sigma,\alpha\mapsto[B]_\sigma}^+$. In other words, $t\rho \in [\![A]\!]_{\sigma,\alpha\mapsto h}^+$ for the particular choice of $h = [B]_\sigma$, and by Ex.3-Q.4, $h \in \mathcal{H}$. So $\langle \_, t\rho\rangle \in [\exists\alpha.A]_\sigma^+$, and since $[\![\exists\alpha.A]\!]_\sigma$ extends $[\exists\alpha.A]_\sigma$ (Ex.3-Q.4), we have $\langle \_, t\rangle\rho = \langle \_, t\rho\rangle \in [\![\exists\alpha.A]\!]_\sigma^+$.

2. [*] Prove *Strong Normalisation*: If $\Gamma \vdash t\!:\!A$ then $t \in \mathsf{SN}$.
   (Hint: choose a valuation σ and a substitution ρ appropriately.)

   Correction : Take σ to map every type variable α to the orthogonal pair $(\mathsf{Var}, \mathsf{Var})$ of simple sets. Take ρ to be the identity substitution mapping every term variable to itself. We have $\rho \in [\![\Gamma]\!]_\sigma$ since for every type B, $[\![B]\!]_\sigma$ is saturated (Ex.3-Q.4), so $\mathsf{Var} \subseteq [\![B]\!]_\sigma^+$, and therefore for every declaration $x\!:\!B$ in Γ, $\rho(x) = x \in [\![B]\!]_\sigma^+$.

   The Adequacy Lemma gives $t\rho \in [\![A]\!]_\sigma^+$, and as $t\rho = t$ and $[\![A]\!]_\sigma^+ \subseteq \mathsf{SN}$ we have $t \in \mathsf{SN}$.