Generic theory combination for model-constructing satisfiability (MCSAT)

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(Ground) Sat-Modulo-Theories problems

Trying to determine whether a collection of formulæ has a model (sat) or not (unsat).
Formulae are here

- built without quantifiers
- defined as terms of sort bool

…terms being those of multi-sorted first-order logic, i.e. built with (free) variables and symbols declared with input and output sorts, e.g.

$$
\begin{aligned}
&f : s_1 \to s_2\\
&+, \times : (\mathrm{Q} \times \mathrm{Q}) \to \mathrm{Q}\\
&\mathrm{is\_prime} : \mathrm{N} \to \mathrm{bool}\\
&=_s : (s \times s) \to \mathrm{bool}\\
&\leq : (\mathrm{Q} \times \mathrm{Q}) \to \mathrm{bool}\\
&\vee, \wedge : (\mathrm{bool} \times \mathrm{bool}) \to \mathrm{bool}\\
&\ldots
\end{aligned}
$$

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The question of satisfiability is asked respectively to a range of theories $\mathcal{T}_1, \ldots, \mathcal{T}_k$, which may impose or restrict the way each sort and each symbol is interpreted:

For instance,

- the Boolean theory imposes that sort Bool be interpreted as $\{\text{true}, \text{false}\}$ and $\vee, \wedge$ be interpreted with the usual truth tables, etc.
- Linear Rational Arithmetic imposes that $+$ be interpreted in the intuitive way, but does not know anything about $\times$, etc

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In presence of other theories, a popular architecture is
DPLL($\bigcup_{i=1}^{n} \mathcal{T}_i$), where

- a front-end is a SAT-solver running DPLL/CDCL;
- it is interfaced with a backend that combines decision procedures for the theories $\mathcal{T}_1, \ldots, \mathcal{T}_n$
(usually by the Nelson-Oppen combination technique)

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Other differences with traditional approaches:

- terms and literals are exchanged that do not belong to the original problem;
- parts that are really specific to the theories can consist of much smaller steps.

1. A glance at MC-Sat

An example in Linear Rational Arithmetic

$l_0 : (-2{\cdot}x - y < 0)$, $l_1 : (x + y < 0)$, $l_2 : (x < -1)$

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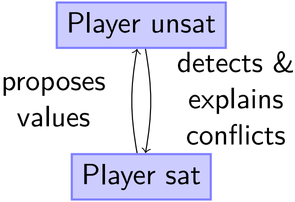
Key ingredients of MC-Sat calculi



Player unsat

proposes values

detects & explains conflicts

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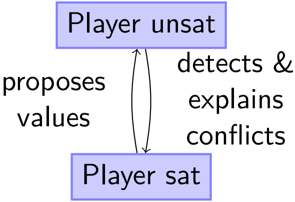

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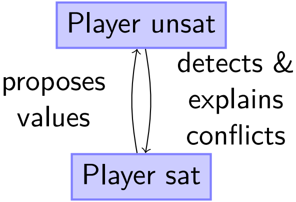

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  With infinite domains (e.g. $\mathbb{Q}$) the "something new" must definitely reject more than 1 value.
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$$(e_1 < x), (x < e_2) \vdash_{\mathsf{LRA}} (e_1 < e_2)$$

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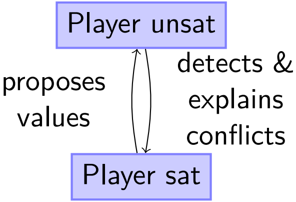

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- Some generic mechanism to expand trails and analyse conflicts

Subtleties

- New literals are introduced during a run (here l_3 and l_4 by FM-resolutions)

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  DPLL's 2-watched literals technique (detecting when to apply Boolean propagation) generalises to n-watched literals & can be used in each theory.

Contributions

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- Is there a way to integrate or generalize both MCSAT and Nelson-Oppen scheme (equality sharing)?

MP Bonacina, N Shankar and SGL address this for disjoint theories in [BGLS16]

2. MC-Sat mechanisms in our formal framework

Same example formalized in our formal framework

Trail = stack of assignments $(t \leftarrow v)$ + "explanation function", initialized with input problem

$(l \leftarrow \text{true})$ abbrev. as l

Empty explanation for input problem

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Level:
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1	$x + y < 0$	$\{\}$	0
2	$x < -1$	$\{\}$	0
3	$y \leftarrow 0$		1

Same example formalized in our formal framework

Trail $=$ stack of assignments $(t \leftarrow v)$ $+$ “explanation function”, initialized with input problem

$(l \leftarrow \text{true})$ abbrev. as l

Empty explanation for input problem

Level:

greatest decision involved

id	trail items	expl.	lev.
0	$-2 \cdot x - y < 0$	$\{\}$	0
1	$x + y < 0$	$\{\}$	0
2	$x < -1$	$\{\}$	0
3	$y \leftarrow 0$		1
4	$-y < -2$	$\{0, 2\}$	0

Same example formalized in our formal framework

Trail = stack of assignments $(t \leftarrow v)$ + "explanation function", initialized with input problem

$(l \leftarrow \text{true})$ abbrev. as l

Empty explanation for input problem

Level:
greatest decision involved

If conflict is of level 0...
...problem is unsat

id	trail items	expl.	lev.
0	$-2 \cdot x - y < 0$	$\{\}$	0
1	$x + y < 0$	$\{\}$	0
2	$x < -1$	$\{\}$	0
3	$\mathbf{y \leftarrow 0}$		1
4	$-\mathbf{y} < -\mathbf{2}$	$\{0, 2\}$	0
	conflict E^1: $\{3, 4\}$		1

Same example formalized in our formal framework

Trail = stack of assignments $(t \leftarrow v)$ + “explanation function”, initialized with input problem

$(l \leftarrow \text{true})$ abbrev. as l
Empty explanation for input problem

Level:
greatest decision involved

If conflict is of level 0...
...problem is unsat

Phase 1

id	trail items	expl.	lev.
0	$-2 \cdot x - y < 0$	$\{\}$	0
1	$x + y < 0$	$\{\}$	0
2	$x < -1$	$\{\}$	0
3	$\mathbf{y \leftarrow 0}$		1
4	$\mathbf{-y < -2}$	$\{0, 2\}$	0
	conflict E^1: $\{3, 4\}$		1

Phase 2

id	trail items	expl.	lev.
0	$-2 \cdot x - y < 0$	$\{\}$	0
1	$x + y < 0$	$\{\}$	0
2	$x < -1$	$\{\}$	0
3	$-y < -2$	$\{0, 2\}$	0

Same example formalized in our formal framework

Trail = stack of assignments $(t \leftarrow v)$ + "explanation function", initialized with input problem

$(l \leftarrow \text{true})$ abbrev. as l

Empty explanation for input problem

Level:

greatest decision involved

If conflict is of level 0...

...problem is unsat

Phase 1

id	trail items	expl.	lev.
0	$-2 \cdot x - y < 0$	$\{\}$	0
1	$x + y < 0$	$\{\}$	0
2	$x < -1$	$\{\}$	0
3	$\mathbf{y \leftarrow 0}$		1
4	$\mathbf{-y < -2}$	$\{0, 2\}$	0
	conflict E^1: $\{3, 4\}$		1

Phase 2

id	trail items	expl.	lev.
0	$-2 \cdot x - y < 0$	$\{\}$	0
1	$x + y < 0$	$\{\}$	0
2	$x < -1$	$\{\}$	0
3	$-y < -2$	$\{0, 2\}$	0
4	$y \leftarrow 4$		1

Same example formalized in our formal framework

Trail = stack of assignments $(t \leftarrow v)$ + "explanation function", initialized with input problem

$(l \leftarrow \text{true})$ abbrev. as l

Empty explanation for input problem

Level:

greatest decision involved

If conflict is of level 0…

…problem is unsat

Phase 1

id	trail items	expl.	lev.
0	$-2 \cdot x - y < 0$	$\{\}$	0
1	$x + y < 0$	$\{\}$	0
2	$x < -1$	$\{\}$	0
3	$\mathbf{y \leftarrow 0}$		1
4	$\mathbf{-y < -2}$	$\{0, 2\}$	0
	conflict E^1: $\{3, 4\}$		1

Phase 2

id	trail items	expl.	lev.
0	$-2 \cdot x - y < 0$	$\{\}$	0
1	$x + y < 0$	$\{\}$	0
2	$x < -1$	$\{\}$	0
3	$-y < -2$	$\{0, 2\}$	0
4	$y \leftarrow 4$		1
5	$y < 0$	$\{0, 1\}$	0

Same example formalized in our formal framework

Trail = stack of assignments $(t \leftarrow v)$ + "explanation function", initialized with input problem

$(l \leftarrow \text{true})$ abbrev. as l

Empty explanation for input problem

Level:

greatest decision involved

If conflict is of level 0...

...problem is unsat

Phase 1

id	trail items	expl.	lev.
0	$-2{\cdot}x - y < 0$	$\{\}$	0
1	$x + y < 0$	$\{\}$	0
2	$x < -1$	$\{\}$	0
3	$\mathbf{y \leftarrow 0}$		1
4	$\mathbf{-y < -2}$	$\{0, 2\}$	0
	conflict E^1: $\{3, 4\}$		1

Phase 2

id	trail items	expl.	lev.
0	$-2{\cdot}x - y < 0$	$\{\}$	0
1	$x + y < 0$	$\{\}$	0
2	$x < -1$	$\{\}$	0
3	$-y < -2$	$\{0, 2\}$	0
4	$\mathbf{y \leftarrow 4}$		1
5	$\mathbf{y < 0}$	$\{0, 1\}$	0
	conflict E^2: $\{4, 5\}$		1

Same example formalized in our formal framework

Trail = stack of assignments $(t \leftarrow v)$ + “explanation function”, initialized with input problem

$(l \leftarrow \text{true})$ abbrev. as l
Empty explanation for input problem

Level:
greatest decision involved

If conflict is of level 0…
…problem is unsat

Phase 1

id	trail items	expl.	lev.
0	$-2{\cdot}x - y < 0$	$\{\}$	0
1	$x + y < 0$	$\{\}$	0
2	$x < -1$	$\{\}$	0
3	$\mathbf{y \leftarrow 0}$		1
4	$\mathbf{-y < -2}$	$\{0, 2\}$	0
	conflict E^1: $\{3, 4\}$		1

Phase 2

id	trail items	expl.	lev.
0	$-2{\cdot}x - y < 0$	$\{\}$	0
1	$x + y < 0$	$\{\}$	0
2	$x < -1$	$\{\}$	0
3	$-y < -2$	$\{0, 2\}$	0
4	$\mathbf{y \leftarrow 4}$		1
5	$\mathbf{y < 0}$	$\{0, 1\}$	0
	conflict E^2: $\{4, 5\}$		1

Phase 3

id	trail items	expl.	lev.
0	$-2{\cdot}x - y < 0$	$\{\}$	0
1	$x + y < 0$	$\{\}$	0
2	$x < -1$	$\{\}$	0
3	$-y < -2$	$\{0, 2\}$	0
4	$y < 0$	$\{0, 1\}$	0

Same example formalized in our formal framework

Trail = stack of assignments $(t \leftarrow v)$ + "explanation function", initialized with input problem

$(l \leftarrow \text{true})$ abbrev. as l

Empty explanation for input problem

Level:

greatest decision involved

If conflict is of level 0. . .

. . . problem is unsat

Phase 1

id	trail items	expl.	lev.
0	$-2 \cdot x - y < 0$	$\{\}$	0
1	$x + y < 0$	$\{\}$	0
2	$x < -1$	$\{\}$	0
3	$\mathbf{y \leftarrow 0}$		1
4	$\mathbf{-y < -2}$	$\{0, 2\}$	0
	conflict E^1: $\{3, 4\}$		1

Phase 2

id	trail items	expl.	lev.
0	$-2 \cdot x - y < 0$	$\{\}$	0
1	$x + y < 0$	$\{\}$	0
2	$x < -1$	$\{\}$	0
3	$-y < -2$	$\{0, 2\}$	0
4	$\mathbf{y \leftarrow 4}$		1
5	$\mathbf{y < 0}$	$\{0, 1\}$	0
	conflict E^2: $\{4, 5\}$		1

Phase 3

id	trail items	expl.	lev.
0	$-2 \cdot x - y < 0$	$\{\}$	0
1	$x + y < 0$	$\{\}$	0
2	$x < -1$	$\{\}$	0
3	$-y < -2$	$\{0, 2\}$	0
4	$y < 0$	$\{0, 1\}$	0
5	$0 < -2$	$\{3, 4\}$	0

Same example formalized in our formal framework

Trail = stack of assignments $(t \leftarrow v)$ + "explanation function", initialized with input problem

$(l \leftarrow \text{true})$ abbrev. as l

Empty explanation for input problem

Level:

greatest decision involved

If conflict is of level 0…

…problem is unsat

Phase 1

id	trail items	expl.	lev.
0	$-2{\cdot}x - y < 0$	$\{\}$	0
1	$x + y < 0$	$\{\}$	0
2	$x < -1$	$\{\}$	0
3	$\mathbf{y \leftarrow 0}$		1
4	$\mathbf{-y < -2}$	$\{0, 2\}$	0
	conflict E^1: $\{3, 4\}$		1

Phase 2

id	trail items	expl.	lev.
0	$-2{\cdot}x - y < 0$	$\{\}$	0
1	$x + y < 0$	$\{\}$	0
2	$x < -1$	$\{\}$	0
3	$-y < -2$	$\{0, 2\}$	0
4	$\mathbf{y \leftarrow 4}$		1
5	$\mathbf{y < 0}$	$\{0, 1\}$	0
	conflict E^2: $\{4, 5\}$		1

Phase 3

id	trail items	expl.	lev.
0	$-2{\cdot}x - y < 0$	$\{\}$	0
1	$x + y < 0$	$\{\}$	0
2	$x < -1$	$\{\}$	0
3	$-y < -2$	$\{0, 2\}$	0
4	$y < 0$	$\{0, 1\}$	0
5	$\mathbf{0 < -2}$	$\{3, 4\}$	0
	conflict E^3: $\{5\}$		0

An example with arithmetic, arrays, congruence

$$f(a[i{:=}\,v][j]) \simeq w\ ,\ w-2 \simeq f(u)\ ,\ i \simeq j\ ,\ u \simeq v$$

Phase 1

id	trail items	expl.	lev.
0	$f(x) \simeq w$	{}	0
1	$y \simeq f(u)$	{}	0
2	$w-2 \simeq y$	{}	0
3	$a[i{:=}\,v][j] \simeq x$	{}	0
4	$i \simeq j$	{}	0
5	$u \simeq v$	{}	0

An example with arithmetic, arrays, congruence

$$f(a[i{:=}\,v][j]) \simeq w\ ,\ \ w-2 \simeq f(u)\ ,\ \ i \simeq j\ ,\ \ u \simeq v$$

Phase 1

id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$y \simeq f(u)$	$\{\}$	0
2	$w-2 \simeq y$	$\{\}$	0
3	$a[i{:=}\,v][j] \simeq x$	$\{\}$	0
4	$i \simeq j$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v{\leftarrow}\mathfrak{c}$		1

An example with arithmetic, arrays, congruence

$$f(a[i{:=}\ v][j]) \simeq w\ ,\ \ w-2 \simeq f(u)\ ,\ \ i \simeq j\ ,\ \ u \simeq v$$

Phase 1

id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$y \simeq f(u)$	$\{\}$	0
2	$w-2 \simeq y$	$\{\}$	0
3	$a[i{:=}\ v][j] \simeq x$	$\{\}$	0
4	$i \simeq j$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v \leftarrow \mathfrak{c}$		1
7	$a[i{:=}\ v][j] \leftarrow \mathfrak{d}$		2

An example with arithmetic, arrays, congruence

$$f(a[i{:=}\ v][j]) \simeq w\ ,\ \ w - 2 \simeq f(u)\ ,\ \ i \simeq j\ ,\ \ u \simeq v$$

	Phase 1		
id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$y \simeq f(u)$	$\{\}$	0
2	$w - 2 \simeq y$	$\{\}$	0
3	$a[i{:=}\ v][j] \simeq x$	$\{\}$	0
4	$i \simeq j$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v{\leftarrow}\mathfrak{c}$		1
7	$a[i{:=}\ v][j]{\leftarrow}\mathfrak{d}$		2
8	$v \not\simeq a[i{:=}\ v][j]$	$\{6, 7\}$	2

An example with arithmetic, arrays, congruence

$$f(a[i:=v][j]) \simeq w\ ,\ w-2 \simeq f(u)\ ,\ i \simeq j\ ,\ u \simeq v$$

	Phase 1		
id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$y \simeq f(u)$	$\{\}$	0
2	$w-2 \simeq y$	$\{\}$	0
3	$a[i:=v][j] \simeq x$	$\{\}$	0
4	$\mathbf{i} \simeq \mathbf{j}$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v \leftarrow \mathfrak{c}$		1
7	$a[i:=v][j] \leftarrow \mathfrak{d}$		2
8	$\mathbf{v} \not\simeq \mathbf{a}[\mathbf{i}:=\mathbf{v}][\mathbf{j}]$	$\{6,7\}$	2
	conflict E^1: $\{4,8\}$		2

An example with arithmetic, arrays, congruence

$$f(a[i{:=}\ v][j]) \simeq w\ ,\ \ w - 2 \simeq f(u)\ ,\ \ i \simeq j\ ,\ \ u \simeq v$$

Phase 1

id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$y \simeq f(u)$	$\{\}$	0
2	$w - 2 \simeq y$	$\{\}$	0
3	$a[i{:=}\ v][j] \simeq x$	$\{\}$	0
4	$\mathbf{i} \simeq \mathbf{j}$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v \leftarrow \mathfrak{c}$		1
7	$a[i{:=}\ v][j] \leftarrow \mathfrak{d}$		2
8	$\mathbf{v} \not\simeq \mathbf{a}[\mathbf{i}{:=}\ \mathbf{v}][\mathbf{j}]$	$\{6, 7\}$	2
	conflict E^1: $\{4, 8\}$		2

Phase 2

id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$y \simeq f(u)$	$\{\}$	0
2	$w - 2 \simeq y$	$\{\}$	0
3	$a[i{:=}\ v][j] \simeq x$	$\{\}$	0
4	$i \simeq j$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v \simeq a[i{:=}\ v][j]$	$\{4\}$	0

An example with arithmetic, arrays, congruence

Phase 2

id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$y \simeq f(u)$	$\{\}$	0
2	$w - 2 \simeq y$	$\{\}$	0
3	$a[i{:=}\ v][j] \simeq x$	$\{\}$	0
4	$i \simeq j$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v \simeq a[i{:=}\ v][j]$	$\{4\}$	0

An example with arithmetic, arrays, congruence

Phase 2

id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$y \simeq f(u)$	$\{\}$	0
2	$w - 2 \simeq y$	$\{\}$	0
3	$a[i{:=}\ v][j] \simeq x$	$\{\}$	0
4	$i \simeq j$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v \simeq a[i{:=}\ v][j]$	$\{4\}$	0
7	$u \leftarrow \mathfrak{c}$		1

An example with arithmetic, arrays, congruence

Phase 2

id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$y \simeq f(u)$	$\{\}$	0
2	$w - 2 \simeq y$	$\{\}$	0
3	$a[i{:=}\ v][j] \simeq x$	$\{\}$	0
4	$i \simeq j$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v \simeq a[i{:=}\ v][j]$	$\{4\}$	0
7	$u \leftarrow \mathfrak{c}$		1
8	$x \leftarrow \mathfrak{c}$		2

An example with arithmetic, arrays, congruence

Phase 2

id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$y \simeq f(u)$	$\{\}$	0
2	$w - 2 \simeq y$	$\{\}$	0
3	$a[i{:=}\ v][j] \simeq x$	$\{\}$	0
4	$i \simeq j$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v \simeq a[i{:=}\ v][j]$	$\{4\}$	0
7	$u {\leftarrow} \mathfrak{c}$		1
8	$x {\leftarrow} \mathfrak{c}$		2
9	$w {\leftarrow} 0$		3

An example with arithmetic, arrays, congruence

Phase 2

id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$y \simeq f(u)$	$\{\}$	0
2	$w - 2 \simeq y$	$\{\}$	0
3	$a[i{:=}\ v][j] \simeq x$	$\{\}$	0
4	$i \simeq j$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v \simeq a[i{:=}\ v][j]$	$\{4\}$	0
7	$u \leftarrow \mathfrak{c}$		1
8	$x \leftarrow \mathfrak{c}$		2
9	$w \leftarrow 0$		3
10	$y \leftarrow -2$		4

An example with arithmetic, arrays, congruence

Phase 2

id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$y \simeq f(u)$	$\{\}$	0
2	$w - 2 \simeq y$	$\{\}$	0
3	$a[i{:=}\ v][j] \simeq x$	$\{\}$	0
4	$i \simeq j$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v \simeq a[i{:=}\ v][j]$	$\{4\}$	0
7	$u \leftarrow \mathfrak{c}$		1
8	$x \leftarrow \mathfrak{c}$		2
9	$w \leftarrow 0$		3
10	$y \leftarrow -2$		4
11	$y \not\simeq w$	$\{9, 10\}$	4

An example with arithmetic, arrays, congruence

Phase 2

id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$y \simeq f(u)$	$\{\}$	0
2	$w - 2 \simeq y$	$\{\}$	0
3	$a[i{:=}\ v][j] \simeq x$	$\{\}$	0
4	$i \simeq j$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v \simeq a[i{:=}\ v][j]$	$\{4\}$	0
7	$u{\leftarrow}\mathfrak{c}$		1
8	$x{\leftarrow}\mathfrak{c}$		2
9	$w{\leftarrow}0$		3
10	$y{\leftarrow} -2$		4
11	$y \not\simeq w$	$\{9, 10\}$	4
12	$u \simeq x$	$\{7, 8\}$	2

An example with arithmetic, arrays, congruence

Phase 2

id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$y \simeq f(u)$	$\{\}$	0
2	$w - 2 \simeq y$	$\{\}$	0
3	$a[i{:=}\ v][j] \simeq x$	$\{\}$	0
4	$i \simeq j$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v \simeq a[i{:=}\ v][j]$	$\{4\}$	0
7	$u \leftarrow \mathfrak{c}$		1
8	$x \leftarrow \mathfrak{c}$		2
9	$w \leftarrow 0$		3
10	$y \leftarrow -2$		4
11	$y \not\simeq w$	$\{9, 10\}$	4
12	$u \simeq x$	$\{7, 8\}$	2
13	$f(u) \simeq f(x)$	$\{12\}$	2

An example with arithmetic, arrays, congruence

Phase 2

id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$y \simeq f(u)$	$\{\}$	0
2	$w - 2 \simeq y$	$\{\}$	0
3	$a[i{:=}\ v][j] \simeq x$	$\{\}$	0
4	$i \simeq j$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v \simeq a[i{:=}\ v][j]$	$\{4\}$	0
7	$u{\leftarrow}\mathfrak{c}$		1
8	$x{\leftarrow}\mathfrak{c}$		2
9	$w{\leftarrow}0$		3
10	$y{\leftarrow} -2$		4
11	$y \not\simeq w$	$\{9, 10\}$	4
12	$u \simeq x$	$\{7, 8\}$	2
13	$f(u) \simeq f(x)$	$\{12\}$	2
14	$f(u) \simeq w$	$\{0, 13\}$	2

An example with arithmetic, arrays, congruence

Phase 2

id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$\mathbf{y} \simeq \mathbf{f}(\mathbf{u})$	$\{\}$	0
2	$w - 2 \simeq y$	$\{\}$	0
3	$a[i{:=}\ v][j] \simeq x$	$\{\}$	0
4	$i \simeq j$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v \simeq a[i{:=}\ v][j]$	$\{4\}$	0
7	$u {\leftarrow} \mathfrak{c}$		1
8	$x {\leftarrow} \mathfrak{c}$		2
9	$w {\leftarrow} 0$		3
10	$y {\leftarrow} -2$		4
11	$\mathbf{y} \not\simeq \mathbf{w}$	$\{9, 10\}$	4
12	$u \simeq x$	$\{7, 8\}$	2
13	$f(u) \simeq f(x)$	$\{12\}$	2
14	$\mathbf{f}(\mathbf{u}) \simeq \mathbf{w}$	$\{0, 13\}$	2
	conflict E^2: $\{1, 11, 14\}$		4

An example with arithmetic, arrays, congruence

Phase 2

id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$\mathbf{y} \simeq \mathbf{f}(\mathbf{u})$	$\{\}$	0
2	$w - 2 \simeq y$	$\{\}$	0
3	$a[i{:=}\ v][j] \simeq x$	$\{\}$	0
4	$i \simeq j$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v \simeq a[i{:=}\ v][j]$	$\{4\}$	0
7	$u{\leftarrow}\mathfrak{c}$		1
8	$x{\leftarrow}\mathfrak{c}$		2
9	$w{\leftarrow}0$		3
10	$y{\leftarrow}-2$		4
11	$\mathbf{y} \not\simeq \mathbf{w}$	$\{9, 10\}$	4
12	$u \simeq x$	$\{7, 8\}$	2
13	$f(u) \simeq f(x)$	$\{12\}$	2
14	$\mathbf{f}(\mathbf{u}) \simeq \mathbf{w}$	$\{0, 13\}$	2
	conflict E^2: $\{1, 11, 14\}$		4

Phase 3

id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$y \simeq f(u)$	$\{\}$	0
2	$w - 2 \simeq y$	$\{\}$	0
3	$a[i{:=}\ v][j] \simeq x$	$\{\}$	0
4	$i \simeq j$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v \simeq a[i{:=}\ v][j]$	$\{4\}$	0
7	$u{\leftarrow}\mathfrak{c}$		1
8	$x{\leftarrow}\mathfrak{c}$		2
9	$u \simeq x$	$\{7, 8\}$	2
10	$f(u) \simeq f(x)$	$\{9\}$	2
11	$f(u) \simeq w$	$\{0, 10\}$	2
12	$y \simeq w$	$\{1, 11\}$	2

An example with arithmetic, arrays, congruence

Phase 3

id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$y \simeq f(u)$	$\{\}$	0
2	$w - 2 \simeq y$	$\{\}$	0
3	$a[i{:=}\ v][j] \simeq x$	$\{\}$	0
4	$i \simeq j$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v \simeq a[i{:=}\ v][j]$	$\{4\}$	0
7	$u{\leftarrow}\mathfrak{c}$		1
8	$x{\leftarrow}\mathfrak{c}$		2
9	$u \simeq x$	$\{7, 8\}$	2
10	$f(u) \simeq f(x)$	$\{9\}$	2
11	$f(u) \simeq w$	$\{0, 10\}$	2
12	$y \simeq w$	$\{1, 11\}$	2

An example with arithmetic, arrays, congruence

Phase 3

id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$y \simeq f(u)$	$\{\}$	0
2	$w - 2 \simeq y$	$\{\}$	0
3	$a[i{:=}\ v][j] \simeq x$	$\{\}$	0
4	$i \simeq j$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v \simeq a[i{:=}\ v][j]$	$\{4\}$	0
7	$u{\leftarrow}\mathfrak{c}$		1
8	$x{\leftarrow}\mathfrak{c}$		2
9	$u \simeq x$	$\{7, 8\}$	2
10	$f(u) \simeq f(x)$	$\{9\}$	2
11	$f(u) \simeq w$	$\{0, 10\}$	2
12	$y \simeq w$	$\{1, 11\}$	2
13	$w - 2 \simeq w$	$\{2, 12\}$	2

An example with arithmetic, arrays, congruence

Phase 3

id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$y \simeq f(u)$	$\{\}$	0
2	$w - 2 \simeq y$	$\{\}$	0
3	$a[i{:=}\ v][j] \simeq x$	$\{\}$	0
4	$i \simeq j$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
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13	$\mathbf{w - 2 \simeq w}$	$\{2, 12\}$	2
	conflict E_1^3: $\{13\}$		2

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13	$w - 2 \simeq w$	$\{2, 12\}$	2
	conflict E_1^3: $\{13\}$		2
	conflict E_2^3: $\{2, 12\}$		2
	conflict E_3^3: $\{1, 2, 11\}$		2

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12	$y \simeq w$	$\{1, 11\}$	2
13	$w - 2 \simeq w$	$\{2, 12\}$	2
	conflict E_1^3: $\{13\}$		2
	conflict E_2^3: $\{2, 12\}$		2
	conflict E_3^3: $\{1, 2, 11\}$		2
	conflict E_4^3: $\{0, 1, 2, 10\}$		2

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1	$\mathbf{y \simeq f(u)}$	$\{\}$	0
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	conflict E_1^3: $\{13\}$		2
	conflict E_2^3: $\{2,12\}$		2
	conflict E_3^3: $\{1,2,11\}$		2
	conflict E_4^3: $\{0,1,2,10\}$		2
	conflict E_5^3: $\{0,1,2,9\}$		2

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	conflict E^3_1: $\{13\}$		2
	conflict E^3_2: $\{2, 12\}$		2
	conflict E^3_3: $\{1, 2, 11\}$		2
	conflict E^3_4: $\{0, 1, 2, 10\}$		2
	conflict E^3_5: $\{0, 1, 2, 9\}$		2

Phase 4

id	trail items	expl.	lev.
0	$f(x) \simeq w$	$\{\}$	0
1	$y \simeq f(u)$	$\{\}$	0
2	$w - 2 \simeq y$	$\{\}$	0
3	$a[i{:=}\ v][j] \simeq x$	$\{\}$	0
4	$i \simeq j$	$\{\}$	0
5	$u \simeq v$	$\{\}$	0
6	$v \simeq a[i{:=}\ v][j]$	$\{4\}$	0
7	$u \not\simeq x$	$\{0, 1, 2\}$	0

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Phase

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0	$\mathbf{f(x) \simeq w}$	$\{\}$	0
1	$\mathbf{y \simeq f(u)}$	$\{\}$	0
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13	$w - 2 \simeq w$	$\{2, 12\}$	2
	conflict E_1^3: $\{13\}$		2
	conflict E_2^3: $\{2, 12\}$		2
	conflict E_3^3: $\{1, 2, 11\}$		2
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7	$u \not\simeq x$	$\{0, 1, 2\}$	0
8	$v \simeq x$	$\{3, 6\}$	0

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8	$\mathbf{v \simeq x}$	$\{3, 6\}$	0
	conflict E^4: $\{5, 7, 8\}$		0

Theory-specific ingredients: $\mathcal{T}$-modules

Given a theory $\mathcal{T}$, a module for $\mathcal{T}$ identifies:

- A collection of sorts for which it will propose values; e.g. sort Q for LRA. These sorts are $\mathcal{T}$-public.

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We add to these inference equality inferences

$$\begin{array}{ll} (t_1 \leftarrow v_1), (t_2 \leftarrow v_2) \vdash t_1 \simeq_s t_2 & \text{if } v_1 \text{ and } v_2 \text{ are the same} \\ (t_1 \leftarrow v_1), (t_2 \leftarrow v_2) \vdash t_1 \not\simeq_s t_2 & \text{if } v_1 \text{ and } v_2 \text{ are different} \end{array}$$

+ reflexivity, symmetry, transitivity.

Example for LRA

LRA-public sorts: just Q.

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$$(e_1 \lessdot_1 x), (x \lessdot_2 e_2) \vdash_{\mathsf{LRA}} (e_1 \lessdot_3 e_2)$$

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- Treatment of disequality:

  $$(e_1 \leq x), (x \leq e_2), (e_1 \simeq e_0), (e_2 \simeq e_0), (\overline{x \simeq e_0}) \vdash_{\mathsf{LRA}} \bot$$

  (triggered only where e_0, e_1 and e_2 have been assigned values)

Design choices

Why make the notion of $\mathcal{T}$-inferences central?

- Rather minimalistic, with derived notions such as: Non-Boolean assignment $(t \leftarrow v)$ "immediately violates" J if there is an inference $J, (t \leftarrow v) \vdash_{\mathcal{T}} L$ with $\overline{L} \in J$

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- Directional (as opposed to, say, a theory lemma):
  premisses of inferences have to be present in the problem,
  conclusion can introduce new material
- Identifies the grains of theory-specific reasoning.
  An MC-Sat derivation of unsat almost explicitly constructs an aggregation of theory inferences
  that can be taken as a proof object (cf. example)

Generic calculus: Search rules

Parameterized by finite set of terms $\mathcal{B}$ called global basis
Let $\mathcal{T}$ be a theory with a specific $\mathcal{T}$-module.

If assignment $t \leftarrow v$ (in $\mathcal{T}$-public sort) does not immediately violate Γ
Decide

$$\Gamma \quad \longrightarrow \quad \Gamma, (t \leftarrow v)$$

If $J \vdash_{\mathcal{T}^=} L$,
with J already in Γ and L is for a formula in $\mathcal{B}$
Propagate

$$\Gamma \quad \longrightarrow \quad \Gamma, (J \vdash L) \quad \text{if } \overline{L} \text{ not in } \Gamma$$

Conflict

$$\Gamma \quad \longrightarrow \quad \Gamma' \qquad \begin{array}{l} \text{if } \overline{L} \text{ in } \Gamma, \\ \text{level}_\Gamma(J, \overline{L}) > 0 \\ \text{and analysing conflict } \langle \Gamma; J, \overline{L} \rangle \text{ gives } \Gamma' \end{array}$$

Fail

$$\Gamma \quad \longrightarrow \quad \text{unsat} \qquad \text{if } \overline{L} \text{ in } \Gamma \text{ and level}_\Gamma(J, \overline{L}) = 0$$

Generic calculus: Conflict analysis rules

Resolve

$\langle \Gamma; E, A \rangle \implies \langle \Gamma; E \cup J \rangle$ if $\mathrm{explain}_\Gamma(A) = J$
& greatest decision in J,
if any, is Boolean

UIPBackjump

$\langle \Gamma; E, L \rangle \implies \Gamma_{\leq \mathrm{level}_\Gamma(E)}, (E \vdash \overline{L})$ if $\mathrm{level}_\Gamma(E) < \mathrm{level}_\Gamma(L)$

SemSplit

$\langle \Gamma; E, L \rangle \implies \Gamma_{\leq \mathrm{level}_\Gamma(L)-1}, \overline{L}$ if $\mathrm{level}_\Gamma(L) = \mathrm{level}_\Gamma(E)$
& there is a decision in $\mathrm{explain}_\Gamma(L)$
& the greatest one is non-Boolean

Undo

$\langle \Gamma; E, A \rangle \implies \Gamma_{\leq \mathrm{level}_\Gamma(A)-1}$ if A is a non-Boolean decision
and $\mathrm{level}_\Gamma(E) < \mathrm{level}_\Gamma(A)$

3. Properties of the calculus

Termination and Soundness

Termination:

If for each theory module $\mathcal{T}$ involved,
there is a local basis $X \mapsto \mathsf{basis}_{\mathcal{T}}(X)$ satisfying some properties,

then it is possible to define a global finite basis for the combination of the theories

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Soundness:

If for each theory module $\mathcal{T}$ involved the $\mathcal{T}$-inferences are sound (i.e. any model endorsing the premisses endorses the conclusion), then if the calculus ends with unsat, then the input was unsat

What happens if we never get unsat?

Do we have a model?

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This relies on a completeness condition for theory modules:
For any Γ,

- Either any model of Γ in the equality theory (where each sort different from bool is interpreted as an infinite countable set) can be extended into a $\mathcal{T}^+$-model of Γ
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Theorem: If all theory modules satisfy the completeness condition, and if the calculus cannot make any further transitions, then the state describes a model.

Proof adapts Nelson-Oppen

Theories for which we provided such theory modules

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$$\begin{aligned}
(t_i \simeq u_i)_{i=1\dots n}, (f(t_1,\dots,t_n) \not\simeq f(u_1,\dots,u_n)) &\vdash_{\mathsf{EUF}} \quad \bot \\
(t_i \simeq u_i)_{i=1\dots n} &\vdash_{\mathsf{EUF}} \quad (f(t_1,\dots,t_n) \simeq f(u_1,\dots,u_n)) \\
(t_i \simeq u_i)_{i=1\dots n, i\neq i_0}, f(t_1,\dots,t_n) \not\simeq f(u_1,\dots,u_n) &\vdash_{\mathsf{EUF}} \quad t_{i_0} \not\simeq u_{i_0}
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(t_i \simeq u_i)_{i=1\dots n, i\neq i_0}, f(t_1,\dots,t_n) \not\simeq f(u_1,\dots,u_n) &\vdash_{\mathsf{EUF}} \quad t_{i_0} \not\simeq u_{i_0}
\end{aligned}$$

- Arrays

$$\begin{aligned}
(t \simeq t'), (i \simeq i'), (t[i] \not\simeq t'[i']) &\vdash_{\mathsf{Arr}} \quad \bot \\
(t \simeq t'), (i \simeq i'), (u \simeq u'), (t[i{:=}\, u] \not\simeq t'[i'{:=}\, u']) &\vdash_{\mathsf{Arr}} \quad \bot \\
(t \simeq t'), (u \simeq u'), (\mathsf{diff}(t,u) \not\simeq \mathsf{diff}(t',u')) &\vdash_{\mathsf{Arr}} \quad \bot \\
(t' \simeq t[i{:=}\, u]), (i \simeq j), (u \not\simeq t'[j]) &\vdash_{\mathsf{Arr}} \quad \bot \\
(t' \simeq t[i{:=}\, u]), (i \not\simeq j), (j \simeq j'), (t[j] \not\simeq t'[j']) &\vdash_{\mathsf{Arr}} \quad \bot \\
(t \not\simeq u) &\vdash_{\mathsf{Arr}} \quad (t[\mathsf{diff}(t,u)] \not\simeq u[\mathsf{diff}(t,u)])
\end{aligned}$$

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- LRA(careful with the local basis)
- EUF

$$\begin{array}{rl} (t_i \simeq u_i)_{i=1\dots n}, (f(t_1, \dots, t_n) \not\simeq f(u_1, \dots, u_n)) \vdash_{\mathsf{EUF}} & \bot \\ (t_i \simeq u_i)_{i=1\dots n} \vdash_{\mathsf{EUF}} & (f(t_1, \dots, t_n) \simeq f(u_1, \dots, u_n)) \\ (t_i \simeq u_i)_{i=1\dots n, i\neq i_0}, f(t_1, \dots, t_n) \not\simeq f(u_1, \dots, u_n) \vdash_{\mathsf{EUF}} & t_{i_0} \not\simeq u_{i_0} \end{array}$$

- Arrays

$$\begin{array}{rl} (t \simeq t'), (i \simeq i'), (t[i] \not\simeq t'[i']) \vdash_{\mathsf{Arr}} & \bot \\ (t \simeq t'), (i \simeq i'), (u \simeq u'), (t[i{:=}\, u] \not\simeq t'[i'{:=}\, u']) \vdash_{\mathsf{Arr}} & \bot \\ (t \simeq t'), (u \simeq u'), (\mathsf{diff}(t, u) \not\simeq \mathsf{diff}(t', u')) \vdash_{\mathsf{Arr}} & \bot \\ (t' \simeq t[i{:=}\, u]), (i \simeq j), (u \not\simeq t'[j]) \vdash_{\mathsf{Arr}} & \bot \\ (t' \simeq t[i{:=}\, u]), (i \not\simeq j), (j \simeq j'), (t[j] \not\simeq t'[j']) \vdash_{\mathsf{Arr}} & \bot \\ (t \not\simeq u) \vdash_{\mathsf{Arr}} & (t[\mathsf{diff}(t, u)] \not\simeq u[\mathsf{diff}(t, u)]) \end{array}$$

- Black box procedure (coarse-grain inferences)

$$l_1 \leftarrow b_1, \dots, l_n \leftarrow b_n \vdash_{\mathcal{T}} \bot$$

where $l_1, \dots, l_n$ are formulæ, and the conjunction of the literals corresponding to the Boolean assignments $l_1 \leftarrow b_1, \dots, l_n \leftarrow b_n$ is $\mathcal{T}$-unsatisfiable

(as detected by e.g. the decision procedure)

Conclusion

In [BGLS16],
We do not assume purification
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Further work:

- non-disjoint theories?
- how to handle quantifiers?
- From proof production to “proved correct” implementation:
  If implementation of each inference is correct and state transitions are correct, then answer is correct
  Separates a kernel that is critical for correctness
  from strategies that is critical for efficiency

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Messages and provability primitives

Implementing this in PSYCHE; each theory $\mathcal{T}$ can emit messages:

`type _ message =`	
`| Unsat : set -> unsat message`	$\Gamma \vdash_{\mathcal{T}} \bot$
`| Infer : set -> form -> infer message`	$\Gamma \vdash_{\mathcal{T}} A$
`| Sat   : set -> sat message`	"$\mathcal{T}$ checks Γ"

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module type Combo = sig
type 'b ans = [...]
val oracle  : 'b message -> 'b ans
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$$\Gamma \vdash_{\mathcal{T}} A \qquad \leadsto \qquad \Gamma \vdash A$$

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resolve

$$\begin{array}{c}\Delta \subseteq \Gamma \\ \Gamma' \subseteq \Gamma\end{array} \qquad \frac{\Delta, l \vdash \bot}{\Gamma \vdash^{?} \bot}\ \Gamma' \vdash l \qquad \rightsquigarrow \qquad \frac{\Delta, l \vdash \bot}{\Delta \cup \Gamma' \vdash \bot}$$

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curryfy

$$\Gamma, A \vdash \bot \quad \rightsquigarrow \quad \Gamma \vdash \neg A$$

Satisfiability primitives

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val sat_init : set -> sat ans
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It means "Γ entirely describes the $\mathcal{T}$-model".

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Here, "$\mathcal{T}$ checks Γ" means **more** than "Γ is $\mathcal{T}$-satisfiable".
It means "Γ entirely describes the $\mathcal{T}$-model".

When no more theories have to check satisfiability of Γ, we stop:
all theories have agreed on model

Trust

```
module type Combo = sig
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Type `'b ans` is private to module `Combo`…
…like type `theorem` of the LCF architecture for theorem proving.

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Here, small steps for theory messages are highly desirable for (2.):
Easier to trust (or prove correct)
the code producing message $(e < x), (x < e') \vdash_{\mathsf{LRA}} (e < e')$
than a full simplex code.