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Call-by-value, call-by-name, and strong normalization for the classical sequent calculus

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Abstract

We present a typed calculus $\lambda\xi$ isomorphic to the implicational fragment of the classical sequent calculus LK. Reductions in LK eliminate the cut-rule by local rewriting steps, which correspond to the evaluation of explicit substitutions in the calculus. This bridges the gap between Curien and Herbelin's $\overline{\lambda}\mu\widetilde{\mu}$-calculus and Urban's rewriting system for proofs. Encodings of one into the other are defined, and from one of them we derive the strong normalization of $\overline{\lambda}\mu\widetilde{\mu}$. Identifying two reduction strategies CBV and CBN in Urban's rewriting system enables us to derive two corresponding semantics of continuations from those of $\overline{\lambda}\mu\widetilde{\mu}$, via the other encoding.

1 Introduction

The sequent calculus, introduced by Gentzen in [Gen35], is a logical system in which the rules only introduce connectives (but on both sides of a sequent), on the contrary to natural deduction which uses introduction and elimination rules. The only way to eliminate a connective is to eliminate the whole formula in which it appears, with an application of the cut-rule. Gentzen calculus for classical logic LK allows sequents of the form $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$, where $A_1, \ldots, A_n$ is to be understood as $A_1 \wedge \ldots \wedge A_n$ and $B_1, \ldots, B_m$ is to be understood as $B_1 \vee .... \vee B_m$. Thus, LK appears as a very symmetrical system.

The cut-rule does not increase the expressive power of the system since a *cut-elimination procedure* has been defined to eliminate all applications of the cut-rule from the proof of a sequent, generating a proof in normal form of the same sequent, that is, with no cut. It proceeds with local rewriting steps of reductions of the proof-tree, and that has the flavour of the evaluation of explicit substitutions, now a wide area of interest in programming theory.

Indeed, the typing rule of an explicit substitution, say in λx [BR95], is nothing else but a cut, and a lot of work has been done to better understand the connection between explicit substitutions and local cut-reduction procedures.

We present here a correspondence *à la* Curry-Howard for LK bringing together the various features of two different approaches that we compare: that of Urban [Urb00] and that of Curien and Herbelin [CH00].

On the logical side, in [Urb00], Urban analyzes thoroughly, among other things, Gentzen-like cut-elimination procedures, and defines a very general reduction system for the proofs in LK which is strongly normalizing, and in which proofs are represented by a syntax of terms. We call the implicational fragment $\lambda\xi$.

Its typing system being isomorphic to LK, the symmetry of the latter naturally leads to the idea of a symmetrical explicit substitution of the form $(M_1\widehat{\alpha} \dagger \widehat{x}M_2)$ typed by the symmetrical cut-rule of LK. The only redexes are the explicit substitutions and the process of cut-elimination corresponds to their evaluation. Normal forms are the terms built from the restricted syntax without explicit substitutions, just as simplified proofs are the proofs built without the cut-rule. The intuitive meaning of the symmetrical substitution would be like "the input x of M_2 is replaced by the output α of M_1" (or is it the output α of M_1 that is replaced by the input x of M_2?).

Such a question tackles the problem of duality in computation between input and output, program and context, or more relevantly between call-by-name and call-by-value evaluation. Indeed, the non-determinism (and non-confluence) of the cut-elimination in LK comes from the presence of many critical pairs, the reductions of which come within the problem of evaluation strategies, which is to be connected to the duality aforementioned. The latter has been shown several times those last few years, in particular the symmetry between CBV and CBN, which Selinger displayed by defining the semantics of Parigot's $\lambda\mu$-calculus ([Par92]) in control and co-control categories [Sel99]. The symmetry is inherent to classical logic, even in natural deduction.

Filinsky had already defined in [Fil89] a symmetrical λ-calculus, based on an approach with continuations, with a nice syntactic conversion of functions as values to deal with higher-order programming. Eventually, Curien and Herbelin managed to connect the duality of computation to the symmetry of classical sequent calculus in a nice way: they define in [CH00] the $\overline{\lambda}\mu\widetilde{\mu}$-calculus to interpret computationally the implicational fragment of LK. Selinger's symmetry gets syntactically exhibited in the $\overline{\lambda}\mu\widetilde{\mu}$-calculus by giving priority to a reduction rule or its dual one.

On the one hand, there is classical logic, and in Parigot's $\lambda\mu$-calculus there is one main conclusion that is being manipulated and possibly several alternative ones. On the other hand, there is sequent calculus and the necessity for the left-introduction rules to manipulate hypotheses, and there the concept of *stoup* that Herbelin has thoroughly studied in [Her95] seems to be much relevant.

One of the key points of $\overline{\lambda}\mu\widetilde{\mu}$-calculus is to notice that the stoup and the main conclusion of $\lambda\mu$ were the dual notions of each other, and to express this duality in a very symmetrical syntax. But the duality goes beyond that: for instance, the symmetry of the reduction rules display syntactically the duality between the CBV and CBN evaluations.

In this paper, we study the relationship between the $\overline{\lambda}\mu\widetilde{\mu}$-calculus and the implicational fragment $\lambda\xi$ of Urban's rewriting system. We encode them into each other. Thus, the former inherits the strong normalization of the latter, and the latter inherits the semantics of the former. In particular, we could not define a denotational semantics of $\lambda\xi$ before we first identified two reduction strategies CBV and CBN in the calculus.

Section 2 presents the classical sequent calculus, section 3 briefly presents the $\overline{\lambda}\mu\widetilde{\mu}$-calculus and its semantics (of continuations). Section 4 presents the implicational fragment of Urban's term rewriting system for LK, as well as the identification of the CBV and CBN evaluation strategies. Section 5 encodes $\overline{\lambda}\mu\widetilde{\mu}$ into Urban's calculus and proves its strong normalization. Section 6 presents the derived semantics for Urban's calculus.

2 Logical setting

We consider the following implicational fragment of Gentzen's system LK for classical logic. The hypotheses and the conclusions of a logical sequent are multi-sets, and the comma will henceforth mean the union of multi-sets.

$$\frac{}{A \vdash A}\mathsf{Ax} \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma' \vdash A, \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathsf{cut}$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', (A \Rightarrow B) \vdash \Delta, \Delta'} \Rightarrow_L \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash (A \Rightarrow B), \Delta} \Rightarrow_R$$

The *structural* rules (contraction and weakening) deal with the multi-sets:

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\mathsf{Contr}_L \quad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\mathsf{Contr}_R \quad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\mathsf{Weak}_L \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\mathsf{Weak}_R$$

Gentzen's cut-elimination procedure eliminates applications of the cut-rule in a proof by moving them upwards, towards the applications of the axiom rule, where they eventually disappear. When we generalize the procedure by allowing the rewriting rules to be applied in any context without restriction, it gives a reduction system which is not deterministic and highly not confluent.

The cut-rule is necessary to simulate the $\Rightarrow$-elimination of natural deduction without having to modify deeply the proofs of the premisses. When λ-calculus (natural deduction) is thus encoded into sequent calculus, if we want to simulate β-reduction by cut-elimination, then some sort of cut-permutation must be allowed. Usually, strong normalization of the cut-elimination is lost when this is allowed. The power of Urban's calculus is its ability to simulate β-reduction without loosing the strong normalization.

3 The $\overline{\lambda}\mu\widetilde{\mu}$-calculus and its semantics

There are two sets of variables: $x, y, z, \ldots$ label the types of the hypotheses and $\alpha, \beta, \gamma, \ldots$ label the types of the conclusions. Moreover the syntax of $\overline{\lambda}\mu\widetilde{\mu}$ has three different categories: commands, terms, and contexts. Correspondingly, they are typed by three kinds of sequents: the usual sequents $\Gamma \vdash \Delta$ type commands, while the sequents typing terms (resp. contexts) are of the form $\Gamma \vdash A|\ \Delta$ (resp. $\Gamma|\ A \vdash \Delta$), making the conclusion (resp. hypothesis) A *active*.

Definition 3.1 [Commands, Terms and Contexts]

$$\begin{array}{ll} c ::= \langle v|e\rangle & \text{(commands)} \\ v ::= x|\mu\beta.c|\lambda x.v & \text{(terms)} \\ e ::= \alpha|\widetilde{\mu}x.c|v \cdot e & \text{(contexts)} \end{array}$$

$\mu\alpha.c$, $\widetilde{\mu}x.c'$ and $\lambda y.v$ respectively bind α in c, x in c' and y in v, and we always consider terms, contexts and commands up to α-conversion, writing them in a way satisfying Barendregt's convention.

A context can be a variable but can also be more complex (so as to have a typing rule introducing $\rightarrow$ on the left-hand side of a sequent), and commands fill the hole of a context with a term. The typing system $\mathsf{LK}_{\mu\widetilde{\mu}}$ is very close to LK:

Definition 3.2 [$\mathsf{LK}_{\mu\widetilde{\mu}}$]

$$\frac{\Gamma \vdash v : A|\Delta \quad \Gamma|e : A \vdash \Delta}{\langle v|e\rangle : (\Gamma \vdash \Delta)}$$ which corresponds to the logical cut-rule

$$\overline{\Gamma|\alpha : A \vdash \alpha : A, \Delta} \quad \overline{\Gamma, x : A \vdash x : A|\Delta}$$

which are two forms corresponding to the logical axiom rule (but with weakening).

$$\frac{\Gamma, x : A \vdash v : B|\Delta}{\Gamma \vdash \lambda x.v : A \rightarrow B|\Delta} \quad \frac{\Gamma \vdash v : A|\Delta \quad \Gamma|e : B \vdash \Delta}{\Gamma|v \cdot e : A \rightarrow B \vdash \Delta}$$

which correspond respectively to the right and left introduction of the arrow.

$$\frac{c : (\Gamma \vdash \beta : A, \Delta)}{\Gamma \vdash \mu\beta.c : A|\Delta} \quad \frac{c : (\Gamma, x : A \vdash \Delta)}{\Gamma|\widetilde{\mu}x.c : A \vdash \Delta}$$

which are two rules specific to the calculus that select the conclusion or the hypothesis that will be manipulated in the next typing rule.

It appears that the contraction and the weakening rules of LK are subtly integrated into the above rules, as it will be discussed in the next section. Note that the type of a context is the type that a term is expected to have in order to fill the hole. With conventional notations about contexts, $v \cdot e$ is to be thought as $e[[\ ]\, v]$. We see here how a term (context) is built either by introducing $\rightarrow$ on the right-hand side (left-hand side) of a sequent, or just by activating one conclusion (hypothesis) from a sequent typing a command: $\mu\alpha.c$ is inherited from Parigot's $\lambda\mu$ [Par92], and $\widetilde{\mu}x.c$ is to be thought as $\mathsf{let}\ x = [\ ]\ \mathsf{in}\ c$. As an example, here is a proof witness for Peirce's Law: $\vdash \lambda z.\mu\alpha.\langle z|(\lambda x.\mu\beta.\langle x|\alpha\rangle)\cdot\alpha\rangle : ((A \rightarrow B) \rightarrow A) \rightarrow A$

Proofs can sometimes be simplified, that is, commands can be computed:

Definition 3.3 [Reductions in $\overline{\lambda}\mu\widetilde{\mu}$]

$$\begin{array}{lrl}
(\rightarrow) & \langle \lambda x.v_1 | v_2 \cdot e\rangle & \rightarrow \langle v_2 | \widetilde{\mu}x.\langle v_1|e\rangle\rangle \\
(\mu) & \langle \mu\beta.c|e\rangle & \rightarrow c[\beta \leftarrow e] \\
(\widetilde{\mu}) & \langle v|\widetilde{\mu}x.c\rangle & \rightarrow c[x \leftarrow v]
\end{array}$$

The system has a critical pair $\langle \mu\alpha.c_1|\ \widetilde{\mu}x.c_2\rangle$ and applying in this case the rule μ gives a call-by-value evaluation, whereas applying the rule $\widetilde{\mu}$ gives a call-by-name evaluation. It is interesting to see that the system with both rules is not confluent, which is connected to the fact that neither is the cut-elimination of the classical sequent calculus.

In [CH00], Curien and Herbelin say that it should be interesting to interpret the typed $\overline{\lambda}\mu\widetilde{\mu}$ in Selinger's control categories. The interpretation of Parigot's $\lambda\mu$ in control categories being based on a semantics of continuations, they define such a semantics for $\overline{\lambda}\mu\widetilde{\mu}$ by translation into CPS. Although the CBN case involves the interpretation of types that Hofmann and Streicher gave in [HS97], we give here as an example an interpretation *à la* Plotkin for its simplicity, as it does not require an extension with products of the lambda-calculus.

Definition 3.4 We define the semantics of terms as follows:

$$\begin{array}{rlrl}
 & \mathsf{CBV}\text{-translation} & & \mathsf{CBN}\text{-translation} \\
\overline{x} & := \lambda k.k\ \widetilde{x} & \underline{x} & := \widetilde{x} \\
\overline{\lambda x.v} & := \lambda k.(k\ (\lambda u.\lambda\widetilde{x}.(\overline{v}\ u))) & \underline{\lambda x.v} & := \lambda k.(k\ \lambda\widetilde{x}.\underline{v}) \\
\overline{\mu\alpha.c} & := \lambda\widetilde{\alpha}.\overline{c} & \underline{\mu\alpha.c} & := \lambda\widetilde{\alpha}.\underline{c} \\
\overline{\alpha} & := \widetilde{\alpha} & \underline{\alpha} & := \lambda u.(u\ \widetilde{\alpha}) \\
\overline{v \cdot e} & := \lambda u.(\overline{v}\ (u\ \overline{e})) & \underline{v \cdot e} & := \lambda u.(c\ (\underline{e}\ \underline{v})) \\
\overline{\widetilde{\mu}x.c} & := \lambda\widetilde{x}.\overline{c} & \underline{\widetilde{\mu}x.c} & := \lambda\widetilde{x}.\underline{c} \\
\overline{\langle v|\ e\rangle} & := \overline{v}\ \overline{e} & \underline{\langle v|\ e\rangle} & := \underline{e}\ \underline{v}
\end{array}$$

As we always want to see the connexion with logic, we are concerned with the translation of types. σ denotes a base type (of $\overline{\lambda}\mu\widetilde{\mu}$), R is the type of responses, V_A denotes a type of values, K_A denotes a type of continuations, and C_A denotes a type of computations, all defined as follows:

Definition 3.5

$$
\begin{array}{ll}
\qquad\textsf{CBV} & \qquad\textsf{CBN} \\
V^{\textsf{CBV}}_{\sigma} := \sigma & V^{\textsf{CBN}}_{\sigma} := \sigma \\
V^{\textsf{CBV}}_{A\to B} := K^{\textsf{CBV}}_{B} \to K^{\textsf{CBV}}_{A} & V^{\textsf{CBN}}_{A\to B} := C^{\textsf{CBN}}_{A} \to C^{\textsf{CBN}}_{B} \\
K^{\textsf{CBV}}_{A} := V^{\textsf{CBV}}_{A} \to R & K^{\textsf{CBN}}_{A} := V^{\textsf{CBN}}_{A} \to R \\
C^{\textsf{CBV}}_{A} := K^{\textsf{CBV}}_{A} \to R & C^{\textsf{CBN}}_{A} := K^{\textsf{CBN}}_{A} \to R
\end{array}
$$

Such an interpretation of types enables us to prove the following:

Theorem 3.6 (Preservation of types)

$$
\begin{cases}
\Gamma \vdash v : A|\Delta \Rightarrow & V^{CBV}_{\Gamma}, K^{CBV}_{\Delta} \vdash \overline{v} : C^{CBV}_{A} \\
 & C^{CBN}_{\Gamma}, K^{CBN}_{\Delta} \vdash \underline{v} : C^{CBN}_{A} \\
\Gamma|\, e : B \vdash \Delta \Rightarrow & V^{CBV}_{\Gamma}, K^{CBV}_{\Delta} \vdash \overline{e} : K^{CBV}_{B} \\
 & C^{CBN}_{\Gamma}, K^{CBN}_{\Delta} \vdash \underline{e} : K^{CBN}_{B} \\
c : (\Gamma \vdash \Delta) \Rightarrow & V^{CBV}_{\Gamma}, K^{CBV}_{\Delta} \vdash \overline{c} : R \\
 & C^{CBN}_{\Gamma}, K^{CBN}_{\Delta} \vdash \underline{c} : R
\end{cases}
$$

The preservation of the types is very interesting from a logical point of view. The typing tree of the semantics is the intuitionistic proof of a sequent in natural deduction, which is the typing system of standard λ-calculus. By modifying the types, we have actually defined two transformations of the logical judgements by adding double negations, making classically provable sequents provable in intuitionistic logic, an idea introduced by Gödel.

Fortunately, it is the case that a CBV-reduction preserves the CBV-semantics and that a CBN-reduction preserves the CBN-semantics (up to α,β-conversion).

Theorem 3.7 (Preservation of semantics)

- *If* $M \to_{\textsf{CBV}} N$ *then* $\overline{M} =_{\alpha,\beta} \overline{N}$
- *If* $M \to_{\textsf{CBN}} N$ *then* $\underline{M} =_{\alpha,\beta} \underline{N}$

Of course, the λ-terms can now be interpreted in a cartesian closed category, and if we compose those interpretations for a command c, we get the morphisms:

$$(V^{\textsf{CBV}}_{A_1} \times \ldots \times V^{\textsf{CBV}}_{A_n} \times K^{\textsf{CBV}}_{B_1} \times \ldots \times K^{\textsf{CBV}}_{B_m}) \xrightarrow{[\![\overline{c}]\!]} R$$

$$(C_{A_1}^{\mathsf{CBN}} \times \ldots \times C_{A_n}^{\mathsf{CBN}} \times K_{B_1}^{\mathsf{CBN}} \times \ldots \times K_{B_m}^{\mathsf{CBN}}) \xrightarrow{[\![c]\!]} R$$

Of course, since we have a translation into CPS, we do not even need a cartesian closed category, we need a response category, which need not have exponential object B^A for every pair A, B, but only for $B = R$, the response object of the category. Besides, we require a response category to have distributive finite products and coproducts and we also require that the morphisms $A \longrightarrow R^{R^A}$ are monic for all A. Note that the sub-category consisting of the objects of the form R^A, called a continuation category, is cartesian closed with: $1 \simeq R^0$, $R^A \times R^B \simeq R^{A+B}$, and $(R^B)^{R^A} \simeq R^{B \times R^A}$, where 1 (resp. 0) is a terminal (resp. initial) object. So we can curryfy the interpretations which become morphisms of that sub-category of continuations:

$$(K_{B_1}^{\mathsf{CBV}} \times \ldots \times K_{B_m}^{\mathsf{CBV}}) \xrightarrow{\Lambda[\![\overline{c}]\!]} R^{V_{A_1}^{\mathsf{CBV}} \times \ldots \times V_{A_n}^{\mathsf{CBV}}}$$

$$(C_{A_1}^{\mathsf{CBN}} \times \ldots \times C_{A_n}^{\mathsf{CBN}}) \xrightarrow{\Lambda[\![c]\!]} R^{K_{B_1}^{\mathsf{CBN}} \times \ldots \times K_{B_m}^{\mathsf{CBN}}}$$

That defines exactly what the semantics in a (co-)control category should be. Indeed, in [Sel99], Selinger proposed two categorical semantics of $\lambda\mu$-calculus: a call-by-name semantics in a control category, and a call-by-value semantics in a co-control category, expressing the duality between the two disciplines. An interesting point of control categories, the definition of which is too long to fit in here, is that every control category is equivalent to a category of continuations. Hence, we have the interpretation of $\overline{\lambda}\mu\widetilde{\mu}$ into a control category for free.

The vantage point of expressing the semantics in a (co-)control category is that the syntax is in a purely logical style (with the disjunction, the conjunction,...) and forgets about the continuations. The computational interpretation of $\overline{\lambda}\mu\widetilde{\mu}$ with continuations where the semantics comes from is completely transparent and the latter appears in a purely algebraic way. As Selinger suggested, we see that in the CBV-semantics should rather be interpreted in a co-control category, since the arrow goes *"the wrong way"*.

$$K_{A_1}^{\mathsf{CBV}} \otimes \ldots \otimes K_{A_n}^{\mathsf{CBV}} \xrightarrow{\Lambda[\![\overline{c}]\!]} K_{B_1}^{\mathsf{CBV}} + \ldots + K_{B_m}^{\mathsf{CBV}}$$

$$C_{A_1}^{\mathsf{CBN}} \times \ldots \times C_{A_n}^{\mathsf{CBN}} \xrightarrow{\Lambda[\![c]\!]} C_{B_1}^{\mathsf{CBN}} \parr \ldots \parr C_{B_m}^{\mathsf{CBN}}$$

where $\parr$ and $\otimes$ are respectively the binoidal functors of the control category and the co-control category.

4 Curry-Howard correspondence for a step-by-step cut-elimination

In the remark 4.1 of [CH00], Curien and Herbelin give a hint on a way to connect $\mathsf{LK}_{\mu\widetilde{\mu}}$ and LK. The proofs of LK embed in $\mathsf{LK}_{\mu\widetilde{\mu}}$ by considering the following sub-syntax of $\overline{\lambda}\mu\widetilde{\mu}$:

$$c ::= \langle x|\ \alpha\rangle|\ \langle \lambda x.\mu\alpha.c|\ \beta\rangle|\ \langle y|\ \mu\alpha.c \cdot \widetilde{\mu}x.c\rangle|\ \langle \mu\alpha.c|\ \widetilde{\mu}x.c\rangle$$

which can be formalized in a simplified syntax, that we shall call (the pure terms of) $\lambda\xi$:

$$M ::= \langle x \cdot \alpha\rangle|\ \widehat{x}M\widehat{\alpha} \cdot \beta|\ M\widehat{\alpha}\,[y]\,\widehat{x}M|\ M\widehat{\alpha} \dagger \widehat{x}M$$

where $\widehat{x}M\widehat{\alpha} \cdot \beta$ binds x and α in M, $M\widehat{\alpha}\,[y]\,\widehat{x}M'$ and $M\widehat{\alpha} \dagger \widehat{x}M'$ bind α in M and x in M'. Again we consider terms up to α-conversion and when we write a term, we always suppose that it satisfies Barendregt's convention. We shall call $^{\mu}$ the obvious embedding of (the pure terms of) $\lambda\xi$ into $\overline{\lambda}\mu\widetilde{\mu}$.

The typing rules are inherited from $\overline{\lambda}\mu\widetilde{\mu}$, and are hence isomorphic to the implicational version of LK with weakening moved in axiom rules and contraction associated to introduction rules. We can always suppose that the constructed formula was already amongst the formulae of the premisses, provided we introduced it at every application of the axiom rule thanks to the weakening.

Definition 4.1 [Typing system of $\lambda\xi$]

$$\frac{}{\langle x \cdot \alpha\rangle : (\Gamma, x : A \vdash \alpha : A, \Delta)}$$

$$\frac{M : (\Gamma, x : A \vdash \alpha : B, \alpha' : A \to B, \Delta)}{(\widehat{x}M\widehat{\alpha} \cdot \alpha') : (\Gamma \vdash \alpha' : A \to B, \Delta)}$$

$$\frac{M : (\Gamma, x' : A \to B \vdash \Delta, \alpha : A) \quad M' : (\Gamma, x' : A \to B, x : B \vdash \Delta)}{M\widehat{\alpha}\,[x']\,\widehat{x}M' : (\Gamma, x' : A \to B \vdash \Delta)}$$

$$\frac{M : (\Gamma \vdash \Delta, \alpha : A) \quad M' : (\Gamma, x : A \vdash \Delta)}{M\widehat{\alpha} \dagger \widehat{x}M' : (\Gamma \vdash \Delta)}$$

The chosen syntax is meant to express an idea that we would like to develop in the future, namely that we like to see the terms as processes and the reductions as communications. Intuitively, the axiom is denoted by $\langle x \cdot \alpha\rangle$, connecting directly an *input channel* x to an *output channel* α. The right introduction is denoted by $\widehat{y}M\widehat{\beta} \cdot \alpha$, meaning that when M receives something (say of type A) on its (private) channel y, it returns something on its (private) channel β (say of type B), thus generating on a specific channel α a transformer from A to B. The left introduction is denoted by $M_1\widehat{\alpha}\,[y]\,\widehat{x}M_2$, meaning that M_1 tries to communicate with M_2, but since α (say of type A) and x (say of type B) do not necessarily fit in their type, a transformer from A to B is expected on channel y to enable the communication. The cut is denoted by $M_1\widehat{\alpha} \dagger \widehat{x}M_2$ and connects the output channel α of M_1 with the input channel x of M_2. In what follows we shall see that it also behaves as an explicit substitution operator.

Unfortunately, the reductions in $\overline{\lambda}\mu\widetilde{\mu}$ do not eliminate every applications of the cut-rule, and that is because the command operator, of which it is the typing rule, has a double role: it either gives to a formula a name that already exists, without creating a redex (typed by a cut which thus simulates a contraction and cannot be eliminated), or creates a redex to be computed (typed by a cut that can be eliminated).

The sub-syntax $\lambda\xi$ constrains the system $\overline{\lambda}\mu\widetilde{\mu}$ by naming every subformula. It is interesting to see how it splits the use of the command. Indeed, the terms of $\lambda\xi$ that are normal forms in $\overline{\lambda}\mu\widetilde{\mu}$ are exactly:

$$M ::= \langle x \cdot \alpha \rangle \,|\, \widehat{x}M\widehat{\alpha} \cdot \beta \,|\, M\widehat{\alpha}\,[y]\,\widehat{x}M$$

where the command is only used to give a name to the formula that has just been introduced. And $M\widehat{\alpha} \dagger \widehat{x}M'$ is always a redex in $\overline{\lambda}\mu\widetilde{\mu}$, and more than a redex, it is a critical pair.

Now this syntax $\lambda\xi$ also appears as a sub-syntax of Urban's proof-describing syntax; so using his terminology, it gives

$$\begin{aligned}
\langle x \cdot \alpha \rangle &= \mathsf{Ax}(x, \alpha) \\
\widehat{x}M\widehat{\alpha} \cdot \beta &= \mathsf{Imp}_R((x)\langle\alpha\rangle M, \beta) \\
M\widehat{\alpha}\,[y]\,\widehat{x}M' &= \mathsf{Imp}_L(\langle\alpha\rangle M, (x)M', y) \\
M\widehat{\alpha} \dagger \widehat{x}M' &= \mathsf{Cut}(\langle\alpha\rangle M, (x)M')
\end{aligned}$$

which defines the *pure* terms, to which he adds $\overleftarrow{\mathsf{Cut}}(\langle\alpha\rangle M, (x)M')$ (now $M\widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x}M'$) and $\overrightarrow{\mathsf{Cut}}(\langle\alpha\rangle M, (x)M')$ (now $M\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}M'$). They are the *active* versions of the (*inactive*) explicit substitution $M\widehat{\alpha} \dagger \widehat{x}M'$, in the sense that they are being computed in a CBV or CBN way, respectively, as we shall see. They are also typed by the cut-rule.
Note that the translation $^{\mu}$ is only defined on pure terms.

Definition 4.2 [Reduction system of $\lambda\xi$]
Now the reduction relation is the closure under any context of the following rules:

Logical rules:

$$\begin{array}{lll}
\langle y\cdot\alpha\rangle\widehat{\alpha}\dagger\widehat{x}\langle x\cdot\beta\rangle & \rightarrow\langle y\cdot\beta\rangle & \\
(\widehat{y}M\widehat{\beta}\cdot\alpha)\widehat{\alpha}\dagger\widehat{x}\langle x\cdot\beta'\rangle & \rightarrow\widehat{y}M\widehat{\beta}\cdot\beta' & \text{if } \alpha\notin FV(M)\\
\langle z\cdot\alpha\rangle\widehat{\alpha}\dagger\widehat{x}(M_1\widehat{\beta}\,[x]\,\widehat{y}M_2) & \rightarrow M_1\widehat{\beta}\,[z]\,\widehat{y}M_2 & \text{if } x\notin FV(M_1)\cup FV(M_2)\\
(\widehat{y}M_1\widehat{\alpha}\cdot\gamma)\widehat{\gamma}\dagger\widehat{z}(M_2\widehat{\beta}\,[z]\,\widehat{x}M_3) & \begin{cases}\rightarrow\ M_2\widehat{\beta}\dagger\widehat{y}(M_1\widehat{\alpha}\dagger\widehat{x}M_3)\\ \rightarrow\ (M_2\widehat{\beta}\dagger\widehat{y}M_1)\widehat{\alpha}\dagger\widehat{x}M_3\end{cases} & \\
 & \text{if } \gamma\notin FV(M_1) \text{ and } z\notin FV(M_2)\cup FV(M_3) &
\end{array}$$

CBV propagation:

$$\begin{array}{rll}
\langle y\cdot\alpha\rangle\widehat{\alpha}\dagger_{\mathsf{V}}\widehat{x}M & \rightarrow\langle y\cdot\alpha\rangle\widehat{\alpha}\dagger\widehat{x}M & \\
\langle y\cdot\beta\rangle\widehat{\alpha}\dagger_{\mathsf{V}}\widehat{x}M & \rightarrow\langle y\cdot\beta\rangle & \text{if } \beta\neq\alpha\\
(\widehat{y}M'\widehat{\beta}\cdot\alpha)\widehat{\alpha}\dagger_{\mathsf{V}}\widehat{x}M & \rightarrow(\widehat{y}(M'\widehat{\alpha}\dagger_{\mathsf{V}}\widehat{x}M)\widehat{\beta}\cdot\alpha)\widehat{\alpha}\dagger\widehat{x}M & \\
(\widehat{y}M'\widehat{\beta}\cdot\gamma)\widehat{\alpha}\dagger_{\mathsf{V}}\widehat{x}M & \rightarrow\widehat{y}(M'\widehat{\alpha}\dagger_{\mathsf{V}}\widehat{x}M)\widehat{\beta}\cdot\gamma & \text{if } \gamma\neq\alpha\\
(M_1\widehat{\beta}\,[z]\,\widehat{y}M_2)\widehat{\alpha}\dagger_{\mathsf{V}}\widehat{x}M & \rightarrow(M_1\widehat{\alpha}\dagger_{\mathsf{V}}\widehat{x}M)\widehat{\beta}\,[z]\,\widehat{y}(M_2\widehat{\alpha}\dagger_{\mathsf{V}}\widehat{x}M) & \\
(M_1\widehat{\beta}\dagger\widehat{y}\langle y\cdot\alpha\rangle)\widehat{\alpha}\dagger_{\mathsf{V}}\widehat{x}M & \rightarrow(M_1\widehat{\alpha}\dagger_{\mathsf{V}}\widehat{x}M)\widehat{\beta}\dagger\widehat{x}M & \\
(M_1\widehat{\beta}\dagger\widehat{y}M_2)\widehat{\alpha}\dagger_{\mathsf{V}}\widehat{x}M & \rightarrow(M_1\widehat{\alpha}\dagger_{\mathsf{V}}\widehat{x}M)\widehat{\beta}\dagger\widehat{y}(M_2\widehat{\alpha}\dagger_{\mathsf{V}}\widehat{x}M) & \text{otherwise}
\end{array}$$

CBN propagation:

$$\begin{array}{lll}
M\widehat{\alpha}\dagger_{\mathsf{N}}\widehat{x}\langle x\cdot\beta\rangle & \rightarrow M\widehat{\alpha}\dagger\widehat{x}\langle x\cdot\beta\rangle & \\
M\widehat{\alpha}\dagger_{\mathsf{N}}\widehat{x}\langle y\cdot\beta\rangle & \rightarrow\langle y\cdot\beta\rangle & \text{if } y\neq x\\
M\widehat{\alpha}\dagger_{\mathsf{N}}\widehat{x}(\widehat{y}M'\widehat{\beta}\cdot\gamma) & \rightarrow\widehat{y}(M\widehat{\alpha}\dagger_{\mathsf{N}}\widehat{x}M')\widehat{\beta}\cdot\gamma & \\
M\widehat{\alpha}\dagger_{\mathsf{N}}\widehat{x}(M_1\widehat{\beta}\,[x]\,\widehat{y}M_2) & \rightarrow M\widehat{\alpha}\dagger\widehat{x}((M\widehat{\alpha}\dagger_{\mathsf{N}}\widehat{x}M_1)\widehat{\beta}\,[x]\,\widehat{y}(M\widehat{\alpha}\dagger_{\mathsf{N}}\widehat{x}M_2)) & \\
M\widehat{\alpha}\dagger_{\mathsf{N}}\widehat{x}(M_1\widehat{\beta}\,[z]\,\widehat{y}M_2) & \rightarrow(M\widehat{\alpha}\dagger_{\mathsf{N}}\widehat{x}M_1)\widehat{\beta}\,[z]\,\widehat{y}(M\widehat{\alpha}\dagger_{\mathsf{N}}\widehat{x}M_2) & \text{if } z\neq x\\
M\widehat{\alpha}\dagger_{\mathsf{N}}\widehat{x}(\langle x\cdot\beta\rangle\widehat{\beta}\dagger\widehat{y}M_2) & \rightarrow M\widehat{\alpha}\dagger\widehat{y}(M\widehat{\alpha}\dagger_{\mathsf{N}}\widehat{x}M_2) & \\
M\widehat{\alpha}\dagger_{\mathsf{N}}\widehat{x}(M_1\widehat{\beta}\dagger\widehat{y}M_2) & \rightarrow(M\widehat{\alpha}\dagger_{\mathsf{N}}\widehat{x}M_1)\widehat{\beta}\dagger\widehat{y}(M\widehat{\alpha}\dagger_{\mathsf{N}}\widehat{x}M_2) & \text{otherwise}
\end{array}$$

Activating the cuts and choosing the strategy:

$$M\widehat{\alpha}\dagger\widehat{x}M'\rightarrow M\widehat{\alpha}\dagger_{\mathsf{V}}\widehat{x}M' \text{ if } M \text{ does not freshly introduce } \alpha$$
$$M\widehat{\alpha}\dagger\widehat{x}M'\rightarrow M\widehat{\alpha}\dagger_{\mathsf{N}}\widehat{x}M' \text{ if } M' \text{ does not freshly introduce } x$$

where M freshly introduces x only if $M\equiv M'\widehat{\beta}\,[x]\,\widehat{y}M''$ with $x\notin FV(M')\cup$

$FV(M'')$ or $M \equiv \langle x \cdot \beta \rangle$ and M freshly introduces α only if $M \equiv \widehat{y}M'\widehat{\beta} \cdot \alpha$ with $\alpha \notin FV(M')$ or $M \equiv \langle y \cdot \alpha \rangle$.

The CBV (resp. CBN) strategy consists in forbidding the CBN-activation (resp. CBV) of an inactive cut when the latter could be CBV-activated (resp. CBN-activated).

Now we see that the reduction rules move the explicit substitution towards the variables of a term step-by-step, and correspondingly, the cut-rule is pushed towards the axiom rules.

This reduction system which we close under context is (a subsystem of) Urban's $(\mathcal{T}^{\leftrightarrow}, \xrightarrow{loc})$ [Urb00].

For instance, in this system, if M is pure (i.e. if it contains no active cut), then

$$M\widehat{\alpha} \dagger \widehat{x}\langle x \cdot \beta \rangle \longrightarrow^* M[\alpha := \beta]$$

$$\langle y \cdot \alpha \rangle \widehat{\alpha} \dagger \widehat{x}M \longrightarrow^* M[x := y]$$

Urban allows those reductions as a one-step rule for any M. But still, a term is a normal form if and only if it does not contain an explicit substitution (which represents, through Curry-Howard correspondence, a proof without cut). Note that M is a normal form in $\lambda\xi$ if and only if M^μ is a normal form in $\overline{\lambda}\mu\widetilde{\mu}$.

The basic property that this reduction system has is the Subject Reduction [Urb00], which is required if we want to see it as proof transformations: the reductions preserve the typing. The proof describes in detail how to move the cut-rule upwards and exhibits explicitly the connection between reductions and cut-elimination.

Urban proved in [Urb00] the strong normalization of $(\mathcal{T}^{\leftrightarrow}, \xrightarrow{loc})$: if a term M is typable, then there is no infinite sequence of reductions starting from it. He first proves the strong normalization of a system with implicit substitutions, adapting the technique of the symmetric reducibility candidates. Using this lemma, he encodes some terms of $\lambda\xi$ into a first-order syntax with a well-founded rpo-order and concludes that any reduction sequence starting from a pure term is finite. Then he generalizes to all the terms by showing that each of them is a reduced form of a suitable pure term (where the cuts have been subtly deactivated). This gives a proof of the cut-elimination in sequent calculus.

5 Strong Normalization of $\overline{\lambda}\mu\widetilde{\mu}$

Although $\lambda\xi$ is a sub-syntax of $\overline{\lambda}\mu\widetilde{\mu}$, $\overline{\lambda}\mu\widetilde{\mu}$ can be encoded into $\lambda\xi$ as follows:

Definition 5.1 [Translation of $\overline{\lambda}\mu\widetilde{\mu}$ into $\lambda\xi$]

$$\langle v|\ e\rangle^\xi := (v)^\xi_\alpha\widehat{\alpha} \dagger \widehat{x}(e)^\xi_x, \text{ where}$$

$(x)^\xi_\alpha \quad := \langle x \cdot \alpha\rangle$	$(\alpha)^\xi_x \quad := \langle x \cdot \alpha\rangle$
$(\lambda x.v)^\xi_\alpha := \widehat{x}(v)^\xi_\beta\widehat{\beta} \cdot \alpha$	$(v \cdot e)^\xi_x := (v)^\xi_\alpha\widehat{\alpha}\, [x]\, \widehat{y}(e)^\xi_y$
$(\mu\beta.c)^\xi_\alpha := (c[\beta := \alpha])^\xi$	$(\widetilde{\mu}y.c)^\xi_x := (c[y := x])^\xi$

Of course, when the translation introduces a new bound variable, we choose a fresh one (ie, not free in the term that we translate). Hence we always suppose $\alpha \notin FV(v)$ when we write $(v)^\xi_\alpha$, and $x \notin FV(e)$ when we write $(e)^\xi_x$. Notice that with this translation, we only get pure terms.

$^\xi$ clearly preserves the type (if $c : (\Gamma \vdash \Delta)$, then $c^\xi : (\Gamma \vdash \Delta)$). Now we can prove that $\lambda\xi$ does implement the notion of substitution in the following sense:

Lemma 5.2

- $(c)^\xi\widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x}(e)^\xi_x \longrightarrow^* (c[\alpha := e])^\xi$
  $(v')^\xi_\beta\widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x}(e)^\xi_x \longrightarrow^* (v'[\alpha := e])^\xi_\beta$ *if* $\beta \notin FV(e)$ *and* $\beta \neq \alpha$
  $(e')^\xi_y\widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x}(e)^\xi_x \longrightarrow^* (e'[\alpha := e])^\xi_y$ *if* $y \notin FV(e)$
- $(v)^\xi_\alpha\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}(c)^\xi \longrightarrow^* (c[x := v])^\xi$
  $(v)^\xi_\alpha\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}(v')^\xi_\beta \longrightarrow^* (v'[x := v])^\xi_\beta$ *if* $\beta \notin FV(v)$
  $(v)^\xi_\alpha\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}(e')^\xi_y \longrightarrow^* (e'[x := v])^\xi_y$ *if* $y \notin FV(v)$ *and* $y \neq x$

Each part can be proved by simultaneous structural induction for the three statements. Using this lemma, we can now prove the simulation of the reductions in $\overline{\lambda}\mu\widetilde{\mu}$.

Theorem 5.3 $c \longrightarrow c'$ *implies* $c^\xi \longrightarrow^+ c'^\xi$.

Proof

- μ-rule: $\langle \mu\alpha.c|e\rangle \rightarrow c[\alpha := e]$. We have $\langle \mu\alpha.c|e\rangle^\xi = (c^\xi)\widehat{\alpha} \dagger \widehat{x}(e)^\xi_x$, which can be activated into $(c^\xi)\widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x}(e)^\xi_x$ because c^ξ does not freshly introduce α (it is a cut). So the previous lemma applies.
- $\widetilde{\mu}$-rule: $\langle v|\widetilde{\mu}x.e\rangle \rightarrow c[x := v]$. Similarly, we get by the previous lemma $\langle v|\widetilde{\mu}x.c\rangle^\xi = (v)^\xi_\alpha\widehat{\alpha} \dagger \widehat{x}(c^\xi) \longrightarrow (v)^\xi_\alpha\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}(c^\xi) \longrightarrow^* (c[x := v])^\xi$.
- $\rightarrow$-rule: $\langle \lambda x.v_1|v_2 \cdot e\rangle \rightarrow \langle v_2|\widetilde{\mu}x.\langle v_1|e\rangle\rangle$.
  We have $\langle \lambda x.v_1|v_2 \cdot e\rangle^\xi = (\widehat{x}(v_1)^\xi_\beta\widehat{\beta} \cdot \alpha)\widehat{\alpha} \dagger \widehat{y}((v_2)^\xi_\gamma\widehat{\gamma}\, [y]\, \widehat{z}(e)^\xi_z)$ and $\langle v_2|\widetilde{\mu}x.\langle v_1|e\rangle\rangle^\xi = (v_2)^\xi_\gamma\widehat{\gamma} \dagger \widehat{x}((v_1)^\xi_\beta\widehat{\beta} \dagger \widehat{z}(e)^\xi_z)$. The former reduces to the latter, since $\widehat{x}(v_1)^\xi_\beta\widehat{\beta} \cdot \alpha$ and $(v_2)^\xi_\gamma\widehat{\gamma}\, [y]\, \widehat{z}(e)^\xi_z$ freshly introduce α and y, respectively.

□

That is all we need to prove the strong normalization of $\overline{\lambda}\mu\widetilde{\mu}$:

Corollary 5.4 *Typed* $\overline{\lambda}\mu\widetilde{\mu}$*-calculus is strongly normalizing.*

Proof Suppose an infinite reduction chain: $c_1 \longrightarrow \cdots \longrightarrow c_i \longrightarrow \cdots$. Using the previous lemma, $c_1^\xi \longrightarrow^+ \cdots \longrightarrow^+ c_i^\xi \longrightarrow^+ \cdots$. Since $^\xi$ preserves the typing and Urban has proved the strong normalization of his system, it is impossible. $\square$

6 Semantics of the step-by-step cut-elimination

We define two sets of terms on which we shall define the CBV and CBN semantics, respectively: $\mathcal{P}_{\mathsf{CBV}}$ is the set of terms in which every CBN-cut $M\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}M'$ is such that M freshly introduces α, and $\mathcal{P}_{\mathsf{CBN}}$ is the set of terms in which every CBV-cut $M\widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x}M'$ is such that M' freshly introduces x.

$\mathcal{P}_{\mathsf{CBV}}$ and $\mathcal{P}_{\mathsf{CBN}}$ both contain the pure terms and are closed under the subterm relation. More important is that $\mathcal{P}_{\mathsf{CBV}}$ is closed under the CBV strategy and $\mathcal{P}_{\mathsf{CBN}}$ is closed under the CBN strategy. Those two sets are intuitively the "well-formed terms with respect to their respective strategy", and are hence the sets of terms bearing a semantical meaning (in a way, they "derive" from pure terms by the CBV and CBN strategies, respectively).

Since the encoding $^\mu$ is only defined on pure terms, we need to define the inactivation $\mathcal{I}(M)$ of a term M as the pure term M in which every active cut (either CBV or CBN) has been replaced by an inactive cut.

Then we define the CBV-translation (resp. CBN-translation) of $\mathcal{P}_{\mathsf{CBV}}$ (resp. $\mathcal{P}_{\mathsf{CBN}}$) into CPS as follows: the interpretation of types is just as for the semantics of $\overline{\lambda}\mu\widetilde{\mu}$, and if $M \in \mathcal{P}_{\mathsf{CBV}}$ (resp. $M \in \mathcal{P}_{\mathsf{CBN}}$), then $\overline{M} := \overline{\mathcal{I}(M)^\mu}$ (resp. $\underline{M} := \underline{\mathcal{I}(M)^\mu}$).

Since $\mathcal{I}$ and $^\mu$ preserve the typing, we have again the preservation of types by the translations:

$$M : (\Gamma \vdash \Delta) \Rightarrow \begin{cases} V_\Gamma^{\mathsf{CBV}}, K_\Delta^{\mathsf{CBV}} \vdash \overline{M} : R \\ C_\Gamma^{\mathsf{CBN}}, K_\Delta^{\mathsf{CBN}} \vdash \underline{M} : R \end{cases}$$

Now the important point is the following:

Theorem 6.1 (Validity with respect to $\overline{\lambda}\mu\widetilde{\mu}$)

- *If $M \in \mathcal{P}_{\mathsf{CBV}}$ and $M \longrightarrow_{\mathsf{CBV}} N$ then $\mathcal{I}(M)^\mu$ and $\mathcal{I}(N)^\mu$ are CBV-joinable.*
- *If $M \in \mathcal{P}_{\mathsf{CBN}}$ and $M \longrightarrow_{\mathsf{CBN}} N$ then $\mathcal{I}(M)^\mu$ and $\mathcal{I}(N)^\mu$ are CBN-joinable.*

This can be proven by giving, for each reduction rule in $\lambda\xi$, the command in $\overline{\lambda}\mu\widetilde{\mu}$ to which $\mathcal{I}(M^\mu)$ and $\mathcal{I}(N^\mu)$ both reduce. This is straightforward but for the CBN-propagation rules in the CBV-strategy and the CBV-propagation rules in the CBN-strategy. In those cases, the hypotheses that the terms are respectively in $\mathcal{P}_{\mathsf{CBV}}$ and $\mathcal{P}_{\mathsf{CBN}}$ are used extensively:

Indeed, a CBN-cut of $\mathcal{P}_{\mathsf{CBV}}$ (resp. a CBV-cut of $\mathcal{P}_{\mathsf{CBN}}$), and hence the term to which it reduces by the corresponding CBN-rule (resp. CBV-rule), are translated into very specific structures of $\overline{\lambda}\mu\widetilde{\mu}$, and for those structures only does the theorem hold.

The key idea is that supposing P freshly introduces α entails $P^\mu = \langle v|\alpha\rangle$ for some v such that $\alpha \notin FV(v)$ and $v \neq \mu\beta.c$, and that supposing P freshly introduces x entails $P^\mu = \langle x|e\rangle$ for some e such that $x \notin FV(e)$ and $e \neq \widetilde{\mu}y.c$.

Hence we can derive from the preservation of the semantics in $\overline{\lambda}\mu\widetilde{\mu}$ the preservation of the semantics of $\lambda\xi$, namely:

Corollary 6.2 (Preservation of the semantics in $\lambda\xi$)

- *If $M \rightarrow_{\mathsf{CBV}} N$ then $\overline{M} =_{\alpha,\beta} \overline{N}$*

- *If $M \rightarrow_{\mathsf{CBN}} N$ then $\underline{M} =_{\alpha,\beta} \underline{N}$*

As another corollary, we also get the validity with respect to the big-step operational semantics:

Corollary 6.3
Supposing the confluence of the CBV and CBN evaluations in $\overline{\lambda}\mu\widetilde{\mu}$, for all pure M and all normal form N,

- *if $M \longrightarrow^*_{\mathsf{CBV}} N$ then $M^\mu \longrightarrow^*_{\mathsf{CBV}} N^\mu$ and N^μ is a normal form*

- *if $M \longrightarrow^*_{\mathsf{CBN}} N$ then $M^\mu \longrightarrow^*_{\mathsf{CBV}} N^\mu$ and N^μ is a normal form.*

This means that the computation of M in $\lambda\xi$ by the CBV-strategy (resp. CBN-strategy) is in fact its CBV-normal form (resp. CBN-normal form) when seen as a term of $\overline{\lambda}\mu\widetilde{\mu}$ (via the translation ${}^\mu$). Note that the confluence of the CBV and CBN evaluations in $\overline{\lambda}\mu\widetilde{\mu}$ is only conjectured, but now that we have the strong normalization of the typed $\overline{\lambda}\mu\widetilde{\mu}$, only the local confluence remains to be proven.

Finally, we can interpret the $\lambda\xi$-calculus categorically just as we did for $\overline{\lambda}\mu\widetilde{\mu}$, by interpreting CPS into a continuation category. We give here as a result what it gives. For CBV:

$$
\begin{array}{ll}
[\![\langle x\cdot\alpha\rangle : (\Gamma, A \vdash A, \Delta)]\!]^{\mathsf{K}}_{\mathsf{V}} & := V_\Gamma \times V_A \times K_A \times K_\Delta \stackrel{\langle \pi_{K_A}, \pi_{V_A}\rangle}{\longrightarrow} K_A \times V_A \stackrel{\epsilon}{\longrightarrow} R \\
[\![\widehat{x}M\widehat{\alpha}\cdot\alpha' : (\Gamma \vdash A \rightarrow B, \Delta)]\!]^{\mathsf{K}}_{\mathsf{V}} & := V_\Gamma \times K_{A\rightarrow B} \times K_\Delta \\
& \stackrel{\mathsf{Id}_{K_{A\rightarrow B}} \times \Lambda[\![M]\!]^{\mathsf{K}}_{\mathsf{V}}}{\longrightarrow} K_{A\rightarrow B} \times R^{V_A \times K_B} \stackrel{\epsilon}{\longrightarrow} R \\
[\![M\widehat{\alpha}\,[x']\,\widehat{x}M' : (\Gamma, A \rightarrow B \vdash \Delta)]\!]^{\mathsf{K}}_{\mathsf{V}} & := V_\Gamma \times V_{A\rightarrow B} \times K_\Delta \\
& \stackrel{\mathsf{Id}_{V_{A\rightarrow B}} \times \langle \mathsf{Id}_{V_\Gamma \times K_\Delta},\, \Lambda[\![M']\!]^{\mathsf{K}}_{\mathsf{V}}\rangle}{\longrightarrow} V_{A\rightarrow B} \times V_\Gamma \times K_\Delta \times R^{V_B} \\
& \stackrel{\mathsf{Id}_{V_\Gamma \times K_\Delta} \times \epsilon}{\longrightarrow} V_\Gamma \times K_\Delta \times K_A \stackrel{[\![M]\!]^{\mathsf{K}}_{\mathsf{V}}}{\longrightarrow} R \\
[\![M\widehat{\alpha} \dagger \widehat{x}M' : (\Gamma \vdash \Delta)]\!]^{\mathsf{K}}_{\mathsf{V}} & := V_\Gamma \times K_\Delta \stackrel{\langle \mathsf{Id},\, \Lambda[\![M']\!]^{\mathsf{K}}_{\mathsf{V}}\rangle}{\longrightarrow} V_\Gamma \times K_\Delta \times R^{V_A} \stackrel{[\![M]\!]^{\mathsf{K}}_{\mathsf{V}}}{\longrightarrow} R
\end{array}
$$

and for CBN:

$$\begin{aligned}
&[\![\langle x\cdot\alpha\rangle : (\Gamma, A\vdash A, \Delta)]\!]^{\mathsf{K}}_{\mathsf{V}} && := C_\Gamma\times C_A\times K_A\times K_\Delta \xrightarrow{\langle\pi_{C_A},\pi_{K_A}\rangle} C_A\times K_A \xrightarrow{\epsilon} R\\
&[\![\widehat{x}M\widehat{\alpha}\cdot\alpha' : (\Gamma\vdash A\to B, \Delta)]\!]^{\mathsf{K}}_{\mathsf{V}} && := C_\Gamma\times K_{A\to B}\times K_\Delta\\
& && \xrightarrow{\langle\pi_{K_{A\to B}},\Lambda[\![M]\!]^{\mathsf{K}}_{\mathsf{N}}\rangle} K_{A\to B}\times R^{K_B\times C_A} \xrightarrow{\epsilon} R\\
&[\![M\widehat{\alpha}\,[x']\,\widehat{x}M' : (\Gamma, A\to B\vdash\Delta)]\!]^{\mathsf{K}}_{\mathsf{V}} && := C_\Gamma\times C_{A\to B}\times K_\Delta \xrightarrow{\langle\pi_{C_{A\to B}},\,\Lambda f\rangle} C_{A\to B}\times R^{V_{A\to B}}\\
& && \xrightarrow{\epsilon} R\\
&[\![M\widehat{\alpha}\dagger\widehat{x}M' : (\Gamma\vdash\Delta)]\!]^{\mathsf{K}}_{\mathsf{V}} && := C_\Gamma\times K_\Delta \xrightarrow{\langle\mathsf{Id},\,\Lambda[\![M]\!]^{\mathsf{K}}_{\mathsf{N}}\rangle} C_\Gamma\times K_\Delta\times R^{K_A} \xrightarrow{[\![M']\!]^{\mathsf{K}}_{\mathsf{N}}} R
\end{aligned}$$

where f is:

$$\begin{aligned}
C_\Gamma\times C_{A\to B}\times K_\Delta\times V_{A\to B} &\xrightarrow{\langle\mathsf{Id},\,\Lambda[\![M]\!]^{\mathsf{K}}_{\mathsf{N}}\rangle\times\mathsf{Id}_{V_{A\to B}}} C_\Gamma\times C_{A\to B}\times K_\Delta\times R^{K_A}\times V_{A\to B}\\
&\xrightarrow{\mathsf{Id}\times\epsilon} C_\Gamma\times C_{A\to B}\times K_\Delta\times C_B \xrightarrow{[\![M']\!]^{\mathsf{K}}_{\mathsf{N}}} R
\end{aligned}$$

And again, we get the semantics in a control category and in a co-control category by curryfying the above semantics as follows:

$$[\![M : (\Gamma\vdash\Delta)]\!]^{\mathsf{C}}_{\mathsf{V}} := \Lambda\left(V_\Gamma\times K_\Delta \xrightarrow{[\![M:(\Gamma\vdash\Delta)]\!]^{\mathsf{K}}_{\mathsf{V}}} R\right) = K_\Delta \xrightarrow{\Lambda[\![M:(\Gamma\vdash\Delta)]\!]^{\mathsf{K}}_{\mathsf{V}}} K_\Gamma$$

$$[\![M : (\Gamma\vdash\Delta)]\!]^{\mathsf{C}}_{\mathsf{N}} := \Lambda\left(C_\Gamma\times K_\Delta \xrightarrow{[\![M:(\Gamma\vdash\Delta)]\!]^{\mathsf{K}}_{\mathsf{N}}} R\right) = C_\Gamma \xrightarrow{\Lambda[\![M:(\Gamma\vdash\Delta)]\!]^{\mathsf{K}}_{\mathsf{N}}} C_\Delta$$

Conclusion

We have managed to encode $\lambda\xi$ into $\overline{\lambda}\mu\widetilde{\mu}$ and vice-versa. We have used one translation to prove the strong normalization of $\overline{\lambda}\mu\widetilde{\mu}$ from that of $\lambda\xi$, and we have identified two strategies in $\lambda\xi$ corresponding to the call-by-name and call-by-value evaluations. Having done so, we were able to give them a denotational semantics of continuations by translating them into CPS and then interpreting CPS into categories. This completes the correspondence *à la* Curry-Howard by connecting the logical features of LK to the computational features of a programming language.

It is likely that the CBV and CBN evaluations are each confluent. This is true when the permutation of cuts is not allowed, by we can only conjecture the property when it is.

The CBN-semantics could also be given using Hofmann and Streicher's interpretation of types, as it seems that a better symmetry with the CBV-semantics should be thus expected. It should also be interesting to extend the semantics to other connectives. It is not difficult to add the constant *false*, or $\perp$, but we would like to investigate what happens with quantifiers.

More generally, in the long term, we would like to work out a system with dependent types, in the aim of reaching a system in the spirit of coq, which, being based on classical sequent calculus and explicit substitutions, could have interesting and efficient features, for instance in proof search.

We would also like to push forward the idea of seeing proof as processes and see to what extent we can thus express the main features of concurrency and mobility.

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