Call-by-value, call-by-name, and strong normalization for the classical sequent calculus

Stéphane Lengrand

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- Calculi for Classical Logic
- $\lambda\xi$-calculus & the strategies
- Encoding into Curien & Herbelin's calculus
- The semantics
- Conclusion and perspectives

Calculi for Classical Logic

CBV, CBN and classical logic

input/output, program/context, value/continuation, CBV/CBN...

- Hofmann, Streicher, Reus (97-98)
  Semantics of continuations (Parigot's $\lambda\mu$).
  Duality CBV/CBN?

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- Selinger (99)
  CBN$\longrightarrow$ Control categories, CBV$\longrightarrow$ Co-Control categories.
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Syntactically?

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Syntactically?

Curien/Herbelin (2000): Yes $\longrightarrow$ the $\overline{\lambda}\mu\widetilde{\mu}$-calculus
(Symmetrical syntax for proof witnesses,
reductions show duality of computation).

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In λx:

$$\frac{\Gamma, (x : B) \vdash M : A \qquad \Gamma \vdash N : B}{\Gamma \vdash M\langle x = N\rangle : A}$$

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... in natural deduction.

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a higher-order reduction system in $\lambda\xi$?

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a higher-order reduction system in $\lambda\xi$?

- Variable-binding
- Local reduction rules. No implicit substitutions.

Our work

	Curien/Herbelin	*Urban*
Curry-Howard for LK		
with a symmetrical syntax	*Yes*	*Yes*
Local reductions to eliminate cuts		
(explicit substitutions)	*No*	*Yes*
CBV/CBN evaluation	*Yes*	*No*
Dual semantics	*Yes*	*No*
Strong Normalization of		
the whole system	?	*Yes*

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The $\lambda\xi$-calculus and the strategies

The classical sequent calculus **LK**

$$\frac{}{A \vdash A}\ \text{axiom}$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', (A \Rightarrow B) \vdash \Delta, \Delta'} \longrightarrow\text{-left} \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash (A \Rightarrow B), \Delta} \longrightarrow\text{-right}$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{cut}$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ \text{contraction-left} \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ \text{contraction-right}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ \text{weakening-left} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\ \text{weakening-right}$$

Dealing with contraction and weakening

$$\dfrac{}{\Gamma, A \vdash A, \Delta}\ \text{axiom}$$

$$\dfrac{\Gamma, (A \Rightarrow B) \vdash A, \Delta \quad \Gamma, (A \Rightarrow B), B \vdash \Delta}{\Gamma, (A \Rightarrow B) \vdash \Delta}\ \longrightarrow\text{-left}$$

$$\dfrac{\Gamma, A \vdash B, (A \Rightarrow B), \Delta}{\Gamma \vdash (A \Rightarrow B), \Delta}\ \longrightarrow\text{-right}$$

$$\dfrac{\Gamma \vdash A, \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\ \text{cut}$$

Syntax and typing system

$$\frac{}{\langle x \cdot \alpha \rangle : (\Gamma, x : A \vdash \alpha : A, \Delta)}$$

$$\frac{M : (\Gamma, x' : A \to B \vdash \Delta, \alpha : A) \quad M' : (\Gamma, x' : A \to B, x : B \vdash \Delta)}{M\widehat{\alpha}\,[x']\,\widehat{x}M' : (\Gamma, x' : A \to B \vdash \Delta)}$$

$$\frac{M : (\Gamma, x : A \vdash \alpha : B, \alpha' : A \to B, \Delta)}{\widehat{x}M\widehat{\alpha} \cdot \alpha' : (\Gamma \vdash \alpha' : A \to B, \Delta)}$$

$$\frac{M : (\Gamma \vdash \Delta, \alpha : A) \quad M' : (\Gamma, x : A \vdash \Delta)}{M\widehat{\alpha} \dagger \widehat{x}M' : (\Gamma \vdash \Delta)}$$

How do we eliminate $M\widehat{\alpha} \dagger \widehat{x}M'$?

- M is an α-value if it freshly introduces α
- M' is an x-covalue if it freshly introduces x

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Types of reductions

- CBV-rules propagate within M until α-value
- CBN-rules propagate within M' until x-covalue
- *Logical* rules: if M α-value and M' an x-covalue, eliminate cut (possibly creating smaller ones)

Logical rules (1):

$$\langle y \cdot \alpha \rangle \widehat{\alpha} \dagger \widehat{x} \langle x \cdot \beta \rangle \quad \rightarrow \langle y \cdot \beta \rangle$$

$$(\widehat{y} M \widehat{\beta} \cdot \alpha) \widehat{\alpha} \dagger \widehat{x} \langle x \cdot \gamma \rangle \quad \rightarrow \widehat{y} M \widehat{\beta} \cdot \gamma$$

$$\text{if } \alpha \notin FV(M)$$

$$\langle z \cdot \alpha \rangle \widehat{\alpha} \dagger \widehat{x} (M_1 \widehat{\beta} [x] \widehat{y} M_2) \quad \rightarrow M_1 \widehat{\beta} [z] \widehat{y} M_2$$

$$\text{if } x \notin FV(M_1) \cup FV(M_2)$$

Logical rules (2):

if $\gamma \notin FV(M_1)$ and $z \notin FV(M_2) \cup FV(M_3)$

$$(\widehat{y}M_1\widehat{\alpha}\cdot\gamma)\widehat{\gamma} \dagger \widehat{z}(M_2\widehat{\beta}\,[z]\,\widehat{x}M_3)$$

$$\swarrow \qquad\qquad\qquad\qquad \searrow$$

$$M_2\widehat{\beta} \dagger \widehat{y}(M_1\widehat{\alpha} \dagger \widehat{x}M_3) \qquad\qquad (M_2\widehat{\beta} \dagger \widehat{y}M_1)\widehat{\alpha} \dagger \widehat{x}M_3$$

CBV propagation:

$$
\begin{array}{rcll}
\langle y \cdot \alpha \rangle \widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x} M & \rightarrow & \langle y \cdot \alpha \rangle \widehat{\alpha} \dagger \widehat{x} M & \\
\langle y \cdot \beta \rangle \widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x} M & \rightarrow & \langle y \cdot \beta \rangle & \text{if } \beta \neq \alpha \\
 & & & \\
(\widehat{y} M' \widehat{\beta} \cdot \alpha) \widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x} M & \rightarrow & (\widehat{y} (M' \widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x} M) \widehat{\beta} \cdot \alpha) \widehat{\alpha} \dagger \widehat{x} M & \\
(\widehat{y} M' \widehat{\beta} \cdot \gamma) \widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x} M & \rightarrow & \widehat{y} (M' \widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x} M) \widehat{\beta} \cdot \gamma & \text{if } \gamma \neq \alpha \\
 & & & \\
(M_1 \widehat{\beta} \, [z] \, \widehat{y} M_2) \widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x} M & \rightarrow & (M_1 \widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x} M) \widehat{\beta} \, [z] \, \widehat{y} (M_2 \widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x} M) & \\
 & & & \\
(M_1 \widehat{\beta} \dagger \widehat{y} \langle y \cdot \alpha \rangle) \widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x} M & \rightarrow & (M_1 \widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x} M) \widehat{\beta} \dagger \widehat{x} M & \\
(M_1 \widehat{\beta} \dagger \widehat{y} M_2) \widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x} M & \rightarrow & (M_1 \widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x} M) \widehat{\beta} \dagger \widehat{y} (M_2 \widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x} M) & \textit{otherwise}
\end{array}
$$

CBN propagation

$$\begin{array}{lcll}
M\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}\langle x \cdot \beta\rangle & \rightarrow & M\widehat{\alpha} \dagger \widehat{x}\langle x \cdot \beta\rangle & \\
M\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}\langle y \cdot \beta\rangle & \rightarrow & \langle y \cdot \beta\rangle & \text{if } y \neq x \\
 & & & \\
M\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}(\widehat{y}M'\widehat{\beta} \cdot \gamma) & \rightarrow & \widehat{y}(M\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}M')\widehat{\beta} \cdot \gamma & \\
 & & & \\
M\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}(M_1\widehat{\beta}\,[x]\,\widehat{y}M_2) & \rightarrow & M\widehat{\alpha} \dagger \widehat{x}((M\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}M_1)\widehat{\beta}\,[x]\,\widehat{y}(M\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}M_2)) & \\
M\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}(M_1\widehat{\beta}\,[z]\,\widehat{y}M_2) & \rightarrow & (M\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}M_1)\widehat{\beta}\,[z]\,\widehat{y}(M\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}M_2) & \text{if } z \neq x \\
 & & & \\
M\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}(\langle x \cdot \beta\rangle\widehat{\beta} \dagger \widehat{y}M_2) & \rightarrow & M\widehat{\alpha} \dagger \widehat{y}(M\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}M_2) & \\
M\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}(M_1\widehat{\beta} \dagger \widehat{y}M_2) & \rightarrow & (M\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}M_1)\widehat{\beta} \dagger \widehat{y}(M\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}M_2) & \mathit{otherwise}
\end{array}$$

Activating the **cuts:**

$$M\widehat{\alpha} \dagger \widehat{x}M'$$

$$\begin{array}{ccc} & & \\ \swarrow & & \searrow \\ M\widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x}M' & & M\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}M' \end{array}$$

M not α-value $\qquad\qquad$ M' not x-covalue

The strategies

When M not an α-value & M' is not an x-covalue,

$M\widehat{\alpha} \dagger \widehat{x}M'$: Unjoinable critical pair

$\Longrightarrow$ irreparable choice: CBV or CBN?

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When M not an α-value & M' is not an x-covalue,
$M\widehat{\alpha} \dagger \widehat{x}M'$: Unjoinable critical pair

$\Longrightarrow$ irreparable choice: CBV or CBN?

- cut-elimination in CBV:
  $M\widehat{\alpha} \dagger \widehat{x}M' \longrightarrow M\widehat{\alpha} \dagger_{\mathsf{N}} \widehat{x}M'$ only if M is an α-value.
  Otherwise you may CBV-activate it and propagate it until it meets one.

- cut-elimination in CBN:
  $M\widehat{\alpha} \dagger \widehat{x}M' \longrightarrow M\widehat{\alpha} \dagger_{\mathsf{V}} \widehat{x}M'$ only if M' is an x-covalue.
  Otherwise you may CBN-activate it and propagate it until it meets one.

Properties

- ***Subject Reduction:***
  If $M \longrightarrow N$, then $M : (\Gamma \vdash \Delta)$ implies $N : (\Gamma \vdash \Delta)$

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  Each evaluation strategy: ?
- Normal form $\Longleftrightarrow$ No cut

- ***Strong Normalization***

If M is typable, then M is strongly normalizing.

Encodings & Semantics

Semantics of $\overline{\lambda}\mu\widetilde{\mu}$

By translation into CPS:

CBV	CBN
$\overline{M}$	$\underline{M}$

Semantics of $\overline{\lambda}\mu\widetilde{\mu}$

By translation into CPS:

$$\begin{array}{cc} \textsf{CBV} & \textsf{CBN} \\ \overline{M} & \underline{M} \end{array}$$

such that

- If $M \longrightarrow_{\textsf{CBV}} N$ then $\overline{M} =_{\alpha,\beta} \overline{N}$
- If $M \longrightarrow_{\textsf{CBN}} N$ then $\underline{M} =_{\alpha,\beta} \underline{N}$

Encodings

$$\lambda\xi \quad \xrightarrow{\mu} \quad \overline{\lambda}\mu\widetilde{\mu}$$

$$\overline{\lambda}\mu\widetilde{\mu} \quad \xrightarrow{\xi} \quad \lambda\xi$$

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$$\lambda\xi \quad \xrightarrow{\mu} \quad \overline{\lambda}\mu\widetilde{\mu}$$

$$\overline{\lambda}\mu\widetilde{\mu} \quad \xrightarrow{\xi} \quad \lambda\xi$$

preserving typing,

such that

$$c \longrightarrow_{\overline{\lambda}\mu\widetilde{\mu}} c' \quad \Rightarrow \quad c^{\mu} \longrightarrow^{*}_{\lambda\xi} c'^{\mu}$$

$\Longrightarrow$ $\overline{\lambda}\mu\widetilde{\mu}$ is strongly normalizing

CBV-Validity

$$M \longrightarrow_{\mathsf{CBV}} M'$$

$$\begin{tikzcd}[column sep=small]
M^{\mu} \arrow[dr, "*\mathsf{CBV}"] & & M'^{\mu} \arrow[dl, "*\mathsf{CBV}"] \\
& c &
\end{tikzcd}$$

$$\overline{M^{\mu}} = \bar{c} = \overline{M'^{\mu}}$$

CBN-Validity

$$M \qquad \longrightarrow_{\mathsf{CBN}} \qquad M'$$

$$\begin{tikzcd}
M^{\mu} \arrow[dr, "*\mathsf{CBN}"] & & M'^{\mu} \arrow[dl, "*\mathsf{CBN}"] \\
& c &
\end{tikzcd}$$

$$\underline{M^{\mu}} \quad = \quad \underline{c} \quad = \quad \underline{M'^{\mu}}$$

Semantics

Semantics by translation into CPS

- **CBV**, *preserved by* **CBV**-*reductions*

$$\begin{cases} V_\sigma & := \widetilde{\sigma} & [\![\langle x|\alpha\rangle]\!]_{\mathsf{V}} & := \widetilde{\alpha}\ \widetilde{x} \\ V_{A\to B} & := K_B \to K_A & [\![\widehat{x}M\widehat{\alpha}\cdot\alpha']\!]_{\mathsf{V}} & := \widetilde{\alpha}'\ (\lambda\widetilde{\alpha}.\lambda\widetilde{x}.\,[\![M]\!]_{\mathsf{V}}) \\ K_A & := V_A \to R & [\![M\widehat{\alpha}\,[x']\,\widehat{x}M']\!]_{\mathsf{V}} & := (\lambda\widetilde{\alpha}.\,[\![M]\!]_{\mathsf{V}})\ (\widetilde{x}'\ (\lambda\widetilde{x}.\,[\![M']\!]_{\mathsf{V}})) \\ C_A & := K_A \to R & [\![M\widehat{\alpha}\dagger\widehat{x}M']\!]_{\mathsf{V}} & := (\lambda\widetilde{\alpha}.\,[\![M]\!]_{\mathsf{V}})\ (\lambda\widetilde{x}.([\![M']\!]_{\mathsf{V}})) \end{cases}$$

- **CBN**, *preserved by* **CBN**-*reductions*

$$\begin{cases} V_\sigma & := \widetilde{\sigma} & [\![\langle x|\alpha\rangle]\!]_{\mathsf{N}} & := \widetilde{x}\ \widetilde{\alpha} \\ V_{A\to B} & := C_A \to C_B & [\![\widehat{x}M\widehat{\alpha}\cdot\alpha']\!]_{\mathsf{N}} & := \widetilde{\alpha}'\ (\lambda\widetilde{x}.\lambda\widetilde{\alpha}.\,[\![M]\!]_{\mathsf{N}}) \\ K_A & := V_A \to R & [\![M\widehat{\alpha}\,[x']\,\widehat{x}M']\!]_{\mathsf{N}} & := \widetilde{x}'\ (\lambda u.((\lambda\widetilde{x}.\,[\![M']\!]_{\mathsf{N}})\ (u\ \lambda\widetilde{\alpha}.\,[\![M]\!]_{\mathsf{N}}))) \\ C_A & := K_A \to R & [\![M\widehat{\alpha}\dagger\widehat{x}M']\!]_{\mathsf{N}} & := (\lambda\widetilde{x}.\,[\![M']\!]_{\mathsf{N}})\ (\lambda\widetilde{\alpha}.([\![M]\!]_{\mathsf{N}})) \end{cases}$$

Semantics in a response category

- **CBV**

  If $M : (x_1 : A_1, \ldots, x_n : A_n \vdash \alpha_1 : B_1, \ldots, \alpha_m : B_m)$ then

  $$\widetilde{x}_1 : V_{A_1}, \ldots, \widetilde{x}_n : V_{A_n}, \widetilde{\alpha}_1 : K_{B_1}, \ldots, \widetilde{\alpha}_m : K_{B_m} \vdash [\![M]\!]_{\mathsf{V}} : R$$

  which corresponds to a morphism

  $$[\![M]\!]^{\mathsf{K}}_{\mathsf{V}} : (V_{A_1} \times \ldots \times V_{A_n} \times K_{B_1} \times \ldots \times K_{B_m}) \to R$$

- **CBN**

  If $M : (x_1 : A_1, \ldots, x_n : A_n \vdash \alpha_1 : B_1, \ldots, \alpha_m : B_m)$ then

  $$\widetilde{x}_1 : C_{A_1}, \ldots, \widetilde{x}_n : C_{A_n}, \widetilde{\alpha}_1 : K_{B_1}, \ldots, \widetilde{\alpha}_m : K_{B_m} \vdash [\![M]\!]_{\mathsf{N}} : R$$

  which corresponds to a morphism

  $$[\![M]\!]^{\mathsf{K}}_{\mathsf{N}} : (C_{A_1} \times \ldots \times C_{A_n} \times K_{B_1} \times \ldots \times K_{B_m}) \to R$$

Semantics in a control category

$\mathcal{C}^c = \{R^A | A \in \mathcal{C}\}$, and $R^A \wp R^B := R^{A \times B}$

- **CBV**

  If

  $$[\![M]\!]^{\mathsf{K}}_{\mathsf{V}} : (V_{A_1} \times \ldots \times V_{A_n} \times K_{B_1} \times \ldots \times K_{B_m}) \to R$$

  then

  $$[\![M]\!]^{\mathsf{C}}_{\mathsf{V}} : K_{B_1} \times \ldots \times K_{B_m} \to K_{A_1} \dagger \ldots \dagger K_{A_n}$$

  or

  $$[\![M]\!]^{\mathsf{C}}_{\mathsf{V}} : K_{A_1} \otimes \ldots \otimes K_{A_n} \to K_{B_1} + \ldots + K_{B_m}$$

- **CBN**

  If

  $$[\![M]\!]^{\mathsf{K}}_{\mathsf{N}} : (C_{A_1} \times \ldots \times C_{A_n} \times K_{B_1} \times \ldots \times K_{B_m}) \to R$$

  then

  $$[\![M]\!]^{\mathsf{C}}_{\mathsf{N}} : C_{A_1} \times \ldots \times C_{A_n} \to C_{B_1} \wp \ldots \wp C_{B_m}$$

Conclusion and perspectives

Conclusion

- A calculus isomorphic to LK, connecting symmetry $\mathsf{CBV}/\mathsf{CBN}$ to that of the classical sequent calculus.
- Encoding into $\overline{\lambda}\mu\widetilde{\mu}$, and back.
- Strong Normalization of $\overline{\lambda}\mu\widetilde{\mu}$
- Two evaluation strategies for cut-elimination in LK
- Semantics of continuations for each of them

Perspectives

- Quantifiers and dependent types?

- Study Hofmann and Streicher's interpretation of types for CBN

- processes: a connection between $M\widehat{\alpha} \dagger \widehat{x}M'$ and the handshake of two programmes M and M'?
  A paradigm proofs-as-processes *à la* Abramsky?

  Types for concurrent/mobile processes bearing logical meaning?

Herbelin's calculus $\overline{\lambda}\mu\widetilde{\mu}$

Terms

$$c ::= \langle v|e \rangle$$

$$v ::= x|\mu\beta.c|\lambda x.v$$

$$e ::= \alpha|\widetilde{\mu}x.c|v \cdot e$$

Reductions

$$\begin{array}{lrcl}
(\rightarrow) & \langle \lambda x.v_{|}v_2 \cdot e \rangle & \rightarrow & \langle v_2|\widetilde{\mu}x.\langle v_{|}e \rangle\rangle \\
(\mu) & \langle \mu\beta.c|e \rangle & \rightarrow & c[\beta := e] \\
(\widetilde{\mu}) & \langle v|\widetilde{\mu}x.c \rangle & \rightarrow & c[x := v]
\end{array}$$

Translation into $\overline{\lambda}\mu\widetilde{\mu}$

$$
\begin{array}{ll}
\langle x|\alpha\rangle^{\mu} & := \langle x|\alpha\rangle \\
(\widehat{x}M\widehat{\alpha}\cdot\beta)^{\mu} & := \langle \lambda x.\mu\alpha.M^{\mu}|\beta\rangle \\
(M\widehat{\alpha}\,[y]\,\widehat{x}M')^{\mu} & := \langle y|(\mu\alpha.M^{\mu})\cdot(\widetilde{\mu}x.M'^{\mu})\rangle \\
(M\widehat{\alpha}\dagger\widehat{x}M')^{\mu} & := \langle \mu\alpha.M^{\mu}|\widetilde{\mu}x.M'^{\mu}\rangle
\end{array}
$$

$$
\begin{array}{rcl}
(x)^{\xi}_{\alpha} & := & \langle x\cdot\alpha\rangle \\
(\lambda x.v)^{\xi}_{\alpha} & := & \widehat{x}(v)^{\xi}_{\beta}\widehat{\beta}\cdot\alpha \\
(\mu\beta.c)^{\xi}_{\alpha} & := & (c[\beta:=\alpha])^{\xi}
\end{array}
$$

$$
\begin{array}{rcl}
(\alpha)^{\xi}_{x} & := & \langle x\cdot\alpha\rangle \\
(v\cdot e)^{\xi}_{x} & := & (v)^{\xi}_{\alpha}\widehat{\alpha}\,[x]\,\widehat{y}(e)^{\xi}_{y} \\
(\widetilde{\mu}y.c)^{\xi}_{x} & := & (c[y:=x])^{\xi}
\end{array}
$$

$$
\begin{array}{ll}
\langle x|\ e\rangle^{\xi} & := (e)^{\xi}_{x} \\
\langle v|\ \alpha\rangle^{\xi} & := (v)^{\xi}_{\alpha} \\
\langle v|\ e\rangle^{\xi} & := (v^{\xi}_{\alpha})\widehat{\alpha}\dagger \\
\text{in every} & \text{other case}
\end{array}
$$