HOW TO PROVE FAITHFULNESS OF ELEMENTARY THEORY INTERPRETATIONS

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ABSTRACT. In an earlier paper, my colleagues and I stated that an interpretation $\mathscr{I}$ of elementary theory Θ in elementary theory Θ' is faithful if and only if, for every model $\mathfrak{A}$ of Θ, there is a model $\mathfrak{A}'$ of Θ' such that the image of $\mathfrak{A}'$ under $\mathscr{I}$ can be expanded to a model of the description of $\mathfrak{A}$. This paper provides a justification of that statement.

In an earlier paper on the foundations of correct architecture refinement [5], my colleagues and I gave a model-theoretic characterization of faithfulness. Although I cannot claim that this characterization is original, I have been unable to find it stated, much less justified, in standard references on model theory [1–4, 6].[1] This paper will provide the justification.[2]

I will assume that the reader is familiar with basic logical concepts, such as *satisfaction*, *model*, *validity*, and *consistency*. Note in particular that a *sentence* is a formula containing no free occurrences of variables, and that a *theory* is a set of sentences that is closed under consequence. The logical notation used below is generally standard or self-explanatory. For example,

$$\mathfrak{A} \vDash \varphi\,[\mathbf{v}_0/a_0, \mathbf{v}_1/a_1, \dots, \mathbf{v}_{n-1}/a_{n-1}],$$

which will usually be shortened to

$$\mathfrak{A} \vDash \varphi\,[a_0, a_1, \dots, a_{n-1}]$$

when the identities of the free variables of φ are unimportant (or made clear by the context), means that the assignment of values $a_0, a_1, \dots, a_{n-1}$ in the domain of the structure $\mathfrak{A}$ to the free variables $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ of the formula φ satisfies φ in $\mathfrak{A}$, while

$$\mathfrak{A} \vDash \varphi(\mathbf{v}_0/\mathbf{t}_0, \mathbf{v}_1/\mathbf{t}_1, \dots, \mathbf{v}_{n-1}/\mathbf{t}_{n-1}),$$

or

$$\mathfrak{A} \vDash \varphi(\mathbf{t}_0, \mathbf{t}_1, \dots, \mathbf{t}_{n-1})$$

for short, means the formula that results when terms $\mathbf{t}_0, \mathbf{t}_1, \dots, \mathbf{t}_{n-1}$ are simultaneously substituted for free occurrences of the variables $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ in the formula φ (renaming bound variables in as necessary to avoid capture of the free

[1] One reference [6, p. 96, Problem 9c] does give a special case as an exercise.

[2] We originally intended to include this argument in an appendix to the earlier paper, but decided that most readers of the journal in which the paper appeared would not find it helpful.

variables of $\mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_{n-1}$) is true in the structure $\mathfrak{A}$. The prefix "$\mathscr{L}$-" will frequently be used to emphasize the elementary language $\mathscr{L}$ under consideration.[3] For example, saying, that φ is an $\mathscr{L}$-formula means that all the parameters of φ are among the parameters of $\mathscr{L}$. I also assume that the reader is familar with the fundamentals of logical metatheory, including the Completeness and Compactness Theorems for first-order logic and the Craig Interpolation Theorem. All this logical background can be found in any good logic textbook, such as the standard model theory references cited above.

For our purposes,[4] it is convenient to consider an interpretation of elementary language $\mathscr{L}$ in elementary theory Θ' to be a mapping $\mathscr{I}$ from $\mathscr{L}$-formulas to $\mathscr{L}'$-formulas that (1) "respects logic", in the sense that, for all $\mathscr{L}$-formulas φ and ψ, and every variable $\mathbf{v}$,

$$\begin{aligned}
\mathscr{I}(\neg\varphi) &= \neg\mathscr{I}(\varphi), \\
\mathscr{I}(\varphi \wedge \psi) &= \mathscr{I}(\varphi) \wedge \mathscr{I}(\psi), \\
\mathscr{I}(\varphi \vee \psi) &= \mathscr{I}(\varphi) \vee \mathscr{I}(\psi), \\
\mathscr{I}(\varphi \rightarrow \psi) &= \mathscr{I}(\varphi) \rightarrow \mathscr{I}(\psi), \\
\mathscr{I}(\forall \mathbf{v}\, \varphi) &= \forall \mathbf{v}\, [\, \omega_{\mathscr{I}}(\mathbf{v}) \rightarrow \mathscr{I}(\varphi)], \text{and} \\
\mathscr{I}(\exists \mathbf{v}\, \varphi) &= \exists \mathbf{v}\, [\, \omega_{\mathscr{I}}(\mathbf{v}) \wedge \mathscr{I}(\varphi)].
\end{aligned}$$

where $\omega_{\mathscr{I}}(\mathbf{v})$ is an $\mathscr{L}'$-formula whose only free variable is $\mathbf{v}$, and (2) satisfies the following conditions:

$$\Theta' \vDash \exists \mathbf{v}\, \omega_{\mathscr{I}}(\mathbf{v}),$$

and, for every individual parameter $\mathbf{a}$ of $\mathscr{L}$,

$$\Theta' \vDash \exists \mathsf{x}_1\, \forall \mathsf{x}_0\, [\, \mathscr{I}(\mathsf{x}_0 \equiv \mathbf{a}) \leftrightarrow \mathsf{x}_0 \equiv \mathsf{x}_1].$$

An interpretation $\mathscr{I}$ of $\mathscr{L}$ in $\mathscr{L}'$-theory Θ' is an *interpretation of* $\mathscr{L}$-theory Θ *in* Θ' iff, for every $\mathscr{L}$-sentence φ,

$$\Theta \vDash \varphi \implies \Theta' \vDash \mathscr{I}(\varphi).$$

A interpretation $\mathscr{I}$ of Θ in Θ' is *faithful* iff, for every $\mathscr{L}$-sentence φ,

$$\Theta' \vDash \mathscr{I}(\varphi) \implies \Theta \vDash \varphi.$$

As a first step toward proving the main result, consider the notion of an *elementary embedding* of one structure in another. For any relational language $\mathscr{L}$, an elementary embedding of $\mathscr{L}$-structure $\mathfrak{A}$ in $\mathscr{L}$-structure $\mathfrak{B}$ is a function f from the domain of $\mathfrak{A}$ to the domain of $\mathfrak{B}$ such that, for any $\mathscr{L}$-formula φ and any assignment of values $a_0, a_1, \ldots, a_{n-1}$ in the domain of $\mathfrak{A}$ to the free variables of φ,

$$\mathfrak{A} \vDash \varphi\, [a_0, a_1, \ldots, a_{n-1}] \quad \text{iff} \quad \mathfrak{B} \vDash \varphi\, [f(a_0), f(a_1), \ldots, f(a_{n-1})]$$

If $\mathfrak{A}$ can be elementarily embedded in $\mathfrak{B}$, then $\mathfrak{A}$ and $\mathfrak{B}$ "agree" on the extension of all formulas. In that case, they certainly agree on the truth value of all sentences, i.e., they are elementarily equivalent.

[3] Since only pure (i.e., non-logical constant-free), function parameter-free, elementary languages will be considered below, languages can be identified with the set of their predicate and individual parameters.

[4] Different authors define *interpretation* somewhat differently. Any of the standard textbook definitions would be adequate for our purposes.

The *description*, or *elementary diagram*, $\mathrm{Ds}(\mathfrak{A})$, of an $\mathscr{L}$-structure $\mathfrak{A}$ with domain A is obtained as follows: expand $\mathscr{L}$ so that it contains a name $\underset{\sim}{a}$ for every member a of A;[5] expand $\mathfrak{A}$ to a structure $(\mathfrak{A}, a)_{a\in A}$ for the language $\mathscr{L} \cup \underset{\sim}{A}$, where $\underset{\sim}{A} = \{\underset{\sim}{a} : a \in A\}$, by assigning each name $\underset{\sim}{a}$ to the corresponding element a; let $\mathrm{Ds}(\mathfrak{A})$ be the set of $\mathscr{L} \cup \underset{\sim}{A}$-sentences true in $(\mathfrak{A}, a)_{a\in A}$. For any $\mathscr{L}$-structure $\mathfrak{A}$, any $\mathscr{L}$-formula φ, and any assignment of values $a_0, a_1, \ldots, a_{n-1}$ in A to the free variables of φ,

$$\mathfrak{A} \vDash \varphi\,[a_0, a_1, \ldots, a_{n-1}] \quad \text{iff} \quad (\mathfrak{A}, a)_{a\in A} \vDash \varphi(\underset{\sim}{a}_0, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_{n-1})$$

So the connection between elementary embeddings and descriptions is straightforward: $\mathfrak{A}$ can be elementarily embedded in $\mathfrak{B}$ iff $\mathfrak{B}$ can be expanded to an $\mathscr{L} \cup \underset{\sim}{A}$-structure that models $\mathrm{Ds}(\mathfrak{A})$.

The bulk of the proof will consist in establishing two lemmas. The first lemma establishes a relationship between *descriptions of structures* and *faithful interpretations.*

Lemma 1: For any faithful interpretation $\mathscr{I}$ of consistent $\mathscr{L}$-theory Θ in $\mathscr{L}'$-theory Θ' and any model $\mathfrak{A}$ of Θ with domain A, the set of sentences $\mathscr{I}[\mathrm{Ds}(\mathfrak{A})] \cup \Theta'$ is consistent, where $\mathscr{I}$ is extended from $\mathscr{L}$ to $\mathscr{L} \cup \underset{\sim}{A}$ by "mapping each new name $\underset{\sim}{a}$ to itself",[6] i.e., for every atomic $\mathscr{L}$-formula φ and every substitution of names $\underset{\sim}{a}_0, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_{n-1}$ in $\underset{\sim}{A}$ for the free variables $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$ of φ, let $\mathscr{I}$ map atomic $\mathscr{L} \cup \underset{\sim}{A}$-formula

$$\varphi(\mathbf{v}_0/\underset{\sim}{a}_0, \mathbf{v}_1/\underset{\sim}{a}_1, \ldots, \mathbf{v}_{n-1}/\underset{\sim}{a}_{n-1})$$

to $\mathscr{L}' \cup \underset{\sim}{A}$-formula

$$\bigwedge_{i<n} \omega_{\mathscr{I}}(\underset{\sim}{a}_i) \wedge \big(\mathscr{I}(\varphi)\big)(\mathbf{v}_0/\underset{\sim}{a}_0, \mathbf{v}_1/\underset{\sim}{a}_1, \ldots, \mathbf{v}_{n-1}/\underset{\sim}{a}_{n-1}).$$

I will first present a proof of this lemma for the special case of conservative extensions — recall that $\mathscr{L}'$-theory Θ' is a conservative extension of $\mathscr{L}$-theory Θ iff the identity mapping on $\mathscr{L}$-sentences is a faithful interpretation of Θ in Θ' — in order to highlight the essential ideas. Then I will explain how the proof is modified to handle other interpretations.

Suppose, for the purpose of deriving a contradiction, that $\mathrm{Ds}(\mathfrak{A}) \cup \Theta'$ is inconsistent. Intuitively, there must be an $\mathscr{L}$-sentence φ such that $\mathrm{Ds}(\mathfrak{A})$ implies that

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[5] Choose fresh names, so that that no $\underset{\sim}{a}$ is in any other language under consideration.

[6] Note that $\varphi(\mathbf{v}_0/\underset{\sim}{a}_0, \mathbf{v}_1/\underset{\sim}{a}_1, \ldots, \mathbf{v}_{n-1}/\underset{\sim}{a}_{n-1})$ is logically equivalent to

$$\exists\mathbf{v}_0\, \exists\mathbf{v}_1 \ldots \exists\mathbf{v} \left[\bigwedge_{i<n} \mathbf{v}_i \equiv \underset{\sim}{a}_i \;\wedge \varphi\right], \tag{1}$$

that formula (1) is mapped to

$$\exists\mathbf{v}_0\, \exists\mathbf{v}_1 \ldots \exists\mathbf{v} \left[\bigwedge_{i<n} \omega_{\mathscr{I}}(\mathbf{v}_i) \wedge \bigwedge_{i<n} \mathbf{v}_i \equiv \underset{\sim}{a}_i \;\wedge \mathscr{I}(\varphi)\right] \tag{2}$$

by $\mathscr{I}$ if every equation $\mathbf{v}_i \equiv \underset{\sim}{a}_i$ is mapped to itself, and that formula (2) is logically equaivalent to

$$\bigwedge_{i<n} \omega_{\mathscr{I}}(\underset{\sim}{a}_i) \wedge (\mathscr{I}(\varphi))(\mathbf{v}_0/\underset{\sim}{a}_0, \mathbf{v}_1/\underset{\sim}{a}_1, \ldots, \mathbf{v}_{n-1}/\underset{\sim}{a}_{n-1}).$$

φ is false and Θ' implies that φ is true. After all, $\mathrm{Ds}(\mathfrak{A})$ says nothing in particular about the additional symbols of $\mathscr{L}'$ that are not in $\mathscr{L}$, and Θ' says nothing in particular about the names $\underset{\sim}{a}$ that were added to $\mathscr{L}$ to obtain $\mathscr{L} \cup \underset{\sim}{A}$; the disagreement between the two theories that is responsible for the inconsistency should be expressible in the common part of their languages. This intuition can be shown to be correct by employing the metatheory of first-order logic. According to the Compactness Theorem, there is a finite subset Δ of $\mathrm{Ds}(\mathfrak{A})$ and a finite subset Γ of Θ' such that $\Delta \cup \Gamma$ is inconsistent. Equivalently,

$$\models \bigwedge \Gamma \rightarrow \neg \bigwedge \Delta$$

By the Completeness Theorem,

$$\vdash \bigwedge \Gamma \rightarrow \neg \bigwedge \Delta$$

Therefore, by the Craig Interpolation Theorem, there is an $\mathscr{L}$-sentence φ such that

$$\vdash \bigwedge \Gamma \rightarrow \varphi \quad \text{and} \quad \vdash \varphi \rightarrow \neg \bigwedge \Delta$$

Equivalently, by the Soundness Theorem,

$$\Gamma \models \varphi \quad \text{and} \quad \Delta \models \neg\varphi$$

and so

$$\varphi \in \Theta' \quad \text{and} \quad \neg\varphi \in \mathrm{Ds}(\mathfrak{A})$$

Since $\mathscr{L}$-sentence φ is in Θ' and the extension is conservative, φ is in Θ. Since $\mathfrak{A}$ is a model of Θ, φ is true in $\mathfrak{A}$ and so is among the sentences in $\mathrm{Ds}(\mathfrak{A})$. But this is impossible, as φ was chosen so that $\neg\varphi$ is in $\mathrm{Ds}(\mathfrak{A})$ and the description of a structure is consistent by definition. Because the supposition that $\mathrm{Ds}(\mathfrak{A}) \cup \Theta'$ is inconsistent led to a contradiction, it follows that $\mathrm{Ds}(\mathfrak{A}) \cup \Theta'$ is consistent.

The proof for interpretations other than the identity is much the same. If we had used the more restrictive notion of *interpretation* where predicate parameters are mapped to predicate parameters and individual parameters to individual parameters [6], essentially the same argument used above would work. But this argument breaks down when our more liberal definition is employed, because interpretations of theories need not be theories, and so a consequence of $\mathscr{I}[\mathrm{Ds}(\mathfrak{A})]$ in which no $\underset{\sim}{a}$ occurs may not be the image of an $\mathscr{L}$-formula under $\mathscr{I}$.[7] However, it is easy to see that if we replace the target language $\mathscr{L}'$ with $\mathscr{L} \uplus \mathscr{L}'$ and also replace $\mathscr{I}[\mathrm{Ds}(\mathfrak{A})]$ by the result of adding both, for every n-ary predicate $\mathbf{P}$ of $\mathscr{L}$, the axiom

$$\forall \mathbf{v}_0 \, \forall \mathbf{v}_1 \ldots \forall \mathbf{v}_{n-1} \, [\, \mathbf{P}(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{n-1}) \leftrightarrow \mathscr{I}(\mathbf{P}(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{n-1}))]$$

and, for every individual parameter $\mathbf{c}$ of $\mathscr{L}$, the axiom

$$\forall \mathbf{v}_0 \, [\, \mathbf{v}_0 \equiv \mathbf{c} \leftrightarrow \mathscr{I}(\mathbf{v}_0 \equiv \mathbf{c})]$$

to a version of $\mathrm{Ds}(\mathfrak{A})$ in which quantifiers are bounded by $\omega_{\mathscr{I}}$, the consistency of the union with Θ' is unaffected. Clearly, the two "say the same thing about the world", in somewhat different languages. So, at least for the purpose of this argument, an

[7] For example, if Θ is the set of consequences of the $\{\mathbf{P}\}$-formula $\forall \mathbf{v} \, \mathbf{P}(\mathbf{v})$ and $\mathscr{I}$ maps $\mathbf{P}(\mathbf{v})$ to $\mathbf{Q}(\mathbf{v}) \wedge \mathbf{R}(\mathbf{v})$, then $\forall \mathbf{v} \, [\omega_{\mathscr{I}}(\mathbf{v}) \rightarrow \mathbf{Q}(\mathbf{v})]$ is a consequence of $\mathscr{I}[\Theta]$ but is not the image of any $\{\mathbf{P}\}$-sentence under $\mathscr{I}$.

interpretation of Θ in Θ' can be viewed as an interpretation in the more restrictive sense of Θ in a definitional extension of Θ'.

The second lemma says that interpretation mappings "preserve meaning".

Lemma 2: For every interpretation $\mathscr{I}$ of $\mathscr{L}$-theory Θ in $\mathscr{L}'$-theory Θ', every model $\mathfrak{A}'$ of Θ', every $\mathscr{L}$ formula φ, and every assignment of values $a_0, a_1, \ldots, a_{n-1}$ in the domain of $\mathfrak{A}'$ to the variables of φ,

$$\mathscr{I}^{\partial}(\mathfrak{A}') \vDash \varphi\,[a_0, a_1, \ldots, a_{n-1}] \quad \text{iff} \quad \mathfrak{A}' \vDash \mathscr{I}(\varphi)\,[a_0, a_1, \ldots, a_{n-1}]$$

where $\mathscr{I}^{\partial}$ is the natural mapping from $\mathscr{L}'$-structures to $\mathscr{L}$-structures determined by $\mathscr{I}$: the domain of $\mathscr{I}^{\partial}(\mathfrak{A}')$ is the same as the domain of $\mathfrak{A}'$, the denotation of n-ary predicate $\mathbf{P}$ of $\mathscr{L}$ in $\mathscr{I}^{\partial}(\mathfrak{A}')$ is

$$\{\langle a_0, a_1, \ldots, a_{n-1}\rangle : \mathfrak{A}' \vDash \mathscr{I}(\mathbf{P}(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{n-1}))\,[a_0, a_1, \ldots, a_{n-1}]\}$$

and the denotation of the individual parameter $\mathbf{c}$ of $\mathscr{L}$ in $\mathscr{I}^{\partial}(\mathfrak{A}')$ is the a such that $\mathfrak{A}' \vDash \mathscr{I}(\mathbf{v} \equiv \mathbf{c})\,[a]$.

Thanks to the absence of function symbols, this lemma can be proved by a straightforward induction on formulas.[8] For atomic formulas, the result is an immediate consequence of the definition of $\mathscr{I}^{\partial}$. If φ is, say, a conjunction $\psi \wedge \chi$, then

$$\begin{aligned}
&\mathscr{I}^{\partial}(\mathfrak{A}') \vDash (\psi \wedge \chi)\,[a_0, a_1, \ldots, a_{n-1}] \\
&\quad \text{iff} \quad \mathscr{I}^{\partial}(\mathfrak{A}') \vDash \psi\,[a_0, a_1, \ldots, a_{n-1}] \text{ and } \mathscr{I}^{\partial}(\mathfrak{A}') \vDash \chi\,[a_0, a_1, \ldots, a_{n-1}] \\
&\qquad\qquad \text{(By the definition of \textit{satisfaction})} \\
&\quad \text{iff} \quad \mathfrak{A}' \vDash \mathscr{I}(\psi)\,[a_0, a_1, \ldots, a_{n-1}] \text{ and } \mathfrak{A}' \vDash \mathscr{I}(\chi)\,[a_0, a_1, \ldots, a_{n-1}] \\
&\qquad\qquad \text{(By the induction hypothesis)} \\
&\quad \text{iff} \quad \mathfrak{A}' \vDash (\mathscr{I}(\psi) \wedge \mathscr{I}(\chi))\,[a_0, a_1, \ldots, a_{n-1}] \\
&\qquad\qquad \text{(By the definition of \textit{satisfaction})} \\
&\quad \text{iff} \quad \mathfrak{A}' \vDash \mathscr{I}(\psi \wedge \chi)\,[a_0, a_1, \ldots, a_{n-1}] \\
&\qquad\qquad \text{(By the definition of \textit{interpretation})}
\end{aligned}$$

The argument for the other connectives and quantifiers is similar.

The main result is an easy consequence of the lemmas.

Theorem: For any interpretation $\mathscr{I}$ of elementary $\mathscr{L}$-theory Θ in elementary $\mathscr{L}'$-theory Θ', the following are equivalent:

(1) $\mathscr{I}$ is faithful;
(2) for every model $\mathfrak{A}$ of Θ there is a model $\mathfrak{A}'$ of Θ' such that $\mathscr{I}^{\partial}(\mathfrak{A}')$ can be expanded to a model of $\mathrm{Ds}(\mathfrak{A})$;
(3) for every model $\mathfrak{A}$ of Θ there is a model $\mathfrak{A}'$ of Θ' such that $\mathfrak{A}$ can be elementarily embedded in $\mathscr{I}^{\partial}(\mathfrak{A}')$.

Proof that (1) implies (2): Let $\mathfrak{A}$ be a model of Θ. According to Lemma 1, $\mathscr{I}[\mathrm{Ds}(\mathfrak{A})] \cup \Theta'$ is consistent. Let $\mathfrak{A}'$ be the reduct of some model $\mathfrak{B}$ of this theory to $\mathscr{L}'$. The structure $\mathscr{I}^{\partial}(\mathfrak{B})$, which is an expansion of $\mathscr{I}^{\partial}(\mathfrak{A}')$ to $\mathscr{L} \cup \underset{\sim}{A}$, is a model of $\mathrm{Ds}(\mathfrak{A})$, by Lemma 2.

[8] Inclusion of function symbols in the language would merely require that the base case of our induction, the atomic formulas, be established by induction on the number of function symbols that occur atoms.

Proof that (2) implies (3): This follows from the connection between elementary embeddings and descriptions noted above.

Proof that (3) implies (1): Let φ be an $\mathscr{L}$-sentence such that $\mathscr{I}(\varphi)$ is in Θ'. Suppose that $\mathfrak{A}$ is a model of Θ. By hypothesis, there is a model $\mathfrak{A}'$ of Θ' such that $\mathfrak{A}$ can be elementarily embedded in $\mathscr{I}^{\partial}(\mathfrak{A}')$. Since $\mathfrak{A}'$ is a model of Θ' and $\mathscr{I}(\varphi)$ is in Θ', $\mathscr{I}(\varphi)$ is true in $\mathfrak{A}'$. But then, using Lemma 2, φ is true in $\mathscr{I}^{\partial}(\mathfrak{A}')$, which is elementarily equivalent to $\mathfrak{A}$. Generalizing, φ is true in every model of Θ, and hence is a member of Θ.

Intuitively, the theorem can be thought of as saying that an interpretation $\mathscr{I}$ of first-order theory Θ in first-order theory Θ' is faithful iff, for every model $\mathfrak{A}$ of Θ, there is a model $\mathfrak{A}'$ of Θ' such that, when the "abstraction map" $\mathscr{I}^{\partial}$ is applied to $\mathfrak{A}'$, the structure that results cannot be distinguished from $\mathfrak{A}$ using the resources of first-order logic. This suggests that analogous results can be established when a different logic is used to formalize the architectures. For example, the logic L^{ω}, often called *higher-order logic* or *simple type theory*, can characterize any structure up to isomorphism.[9] One would therefore expect that an interpretation $\mathscr{I}$ of an L^{ω} theory Θ in an L^{ω} theory Θ' is faithful iff, for every model $\mathfrak{A}$ of Θ, there is a model $\mathfrak{A}'$ of Θ' such that $\mathscr{I}^{\partial}(\mathfrak{A}')$ is isomorphic to $\mathfrak{A}$. It is easy to prove that this expectation is correct. Formulating and proving analogues of the theorem for other logics of interest is left as an exercise.

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[9] L^{ω} allows quantification over relations among individuals (or, equivalently, functions from individuals to individuals), relations among relations among individuals (or functions from functions on individuals to functions on individuals), and so on.