Uniform Description of Efficient Decision Procedures Using Extended Signatures

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Formal Logic

Formal logic provides languages for precise formulation of

- specification of hardware and software designs, protocols,
- correctness properties of programs,
- queries for database search, . . .

Inference mechanisms help to

- prove correctness of specifications,
- answer search queries, . . .

Formal Logic

Applications usually involve many different "theories".

For example, in a typical program correctness application, if,

$$\begin{array}{rcl} \textbf{select}(v, i) & : & \text{Select the } i\text{-th element of array } v \\ \textbf{store}(v, i, e) & : & \text{Store } e \text{ as the } i\text{-th element of array } v \end{array}$$

then, we may obtain:

$$\begin{array}{rcl} \phi & : & \textbf{select}(\textbf{store}(v, i, e+1), i) \approx [1 + \textbf{select}(v, i)] \wedge \\ & & (e < \textbf{select}(v, i)) \end{array}$$

Theorem Proving

General theorem provers are systems that automatically infer new facts from known ones.

How to handle applications involving different “theories”?
Add axioms corresponding to various useful theories into a general-purpose deduction system.

But...theorem provers are:

$+$: Powerful (expressiveness) $-$: Unpredictable and slow

and adding more axioms does *not* help!

Decision Procedures

Decision procedures are specialized procedures designed for subclass of formulas possibly from a particular domain.

$-$: Limited in use $\qquad$ $+$: Fast, predictable

Examples include

- algorithms in a computer algebra system
- Presburger arithmetic
- congruence closure
- model checking, ...

The Combination Problem

$T_1,\ T_2$	:	FO theories over signatures Σ_1, Σ_2.
$T = T_1 \cup T_2$	:	FO theory over $\Sigma_1 \cup \Sigma_2$.
$\Pi(T)$	:	Satisfiability problem of quantifier-free formulas in theory T.

Given decision procedures for $\Pi(T_1)$ and $\Pi(T_2)$, can we get one for $\Pi(T)$?

Variable Abstraction

Terms	$t[s]$	$\mapsto$	$t[x]$	$\wedge$	$x \approx s$
Formulae	ϕ	$\mapsto$	ϕ_1	$\wedge$	ϕ_2
	$(\Sigma_1 \cup \Sigma_2)$		$(\Sigma_1 \cup \mathcal{V})$		$(\Sigma_2 \cup \mathcal{V})$

Example:

$$\begin{array}{lcl}
\phi & : & \textbf{select}(\textbf{store}(v, i, e+1), i) \approx [1 + \textbf{select}(v, i)] \wedge \\
 & & (e < \textbf{select}(v, i)) \\
\phi_1 & : & \textbf{select}(\textbf{store}(v, i, z_0), i) \approx z_1 \ \wedge (\textbf{select}(v, i) \approx z_2) \\
\phi_2 & : & (z_0 \approx e + 1) \ \wedge \ (1 + z_2 \approx z_1) \ \wedge \ (e < z_2)
\end{array}$$

Congruence Closure: Problem

Σ : Signature containing constants and function symbols

$\Sigma = \{f_1, f_2, f_3, \ldots, f_n\}$

ϕ : $t_1 \approx s_1 \;\wedge\; t_2 \approx s_2 \;\wedge \cdots \wedge\; t_k \approx s_k \;\wedge$

$(t'_1 \not\approx s'_1 \;\wedge \cdots \wedge\; t'_l \neq s'_l)$

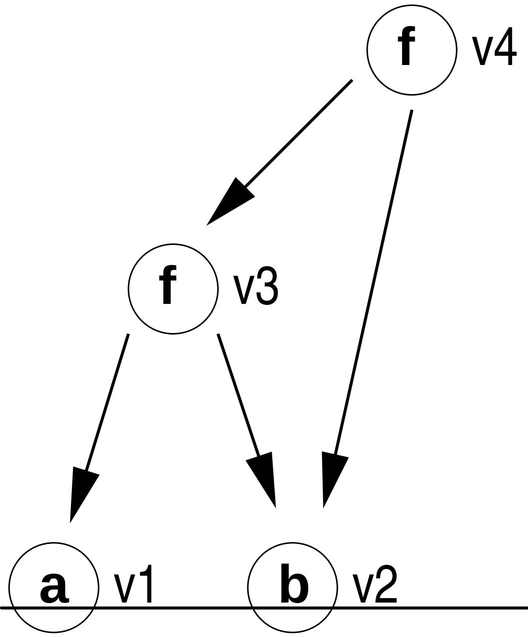
t_i, s_i, t'_i, s'_i are terms over Σ

Is ϕ satisfiable?

Compute congruence closure of the equations and check for each inequality.

Example of Congruence Closure

If $\mathcal{E} = \{f(f(a,b),b) \approx b,\ f(a,b) \approx a\}$, then the terms are represented by the DAG below, and:



$$G_{\mathcal{E}} = \{(v2, v3), (v1, v4)\}.$$

The final congruence closure is

$$\{\{v1, v2, v3, v4\}\}.$$

Congruence Closure: A Different Look

T_i : Theory of equality over $\Sigma_i = \{f_i\}$ and constants U

$$\phi \quad : \quad t_1 \approx s_1 \ \wedge \ t_2 \approx s_2 \ \wedge \cdots \wedge \ t_k \approx s_k \ \wedge$$
$$(t'_1 \not\approx s'_1 \ \wedge \cdots \wedge \ t'_l \neq s'_l)$$

s_i, t_i, s'_i, t'_i terms over $\cup_i \Sigma_i$

Is ϕ satisfiable?

Use variable abstraction and deal with T_i separately!

Signatures and Extensions

How do equations look like?

D-rules : $f(c_1, \ldots, c_k) \rightarrow c_0$, where $f \in \Sigma$ and $c_0, \ldots, c_k \in U$.

C-rules : $c \rightarrow d$, where $c, d \in U$.

Example. Let $\Sigma_0 = \{a,\ b,\ f\}$, and let

$$\mathcal{E}_0 = \{f(f(a,b),b) \approx b,\quad f(a,b) \approx a\}.$$

Then,

$$\begin{aligned} D_0 &= \{a \rightarrow c_1,\ b \rightarrow c_2,\ f(c_1,c_2) \rightarrow c_3,\ f(c_3,c_2) \rightarrow c_4\}, \\ C_0 &= \{c_4 \rightarrow c_2,\ c_3 \rightarrow c_1\}. \end{aligned}$$

Here, $U = \{c_1, c_2, \ldots\}$.

Individual Theories

How to reason in T_i?

Use standard critical-pair completion for ground equations over $\Sigma_i \cup U$.

Superposition: $$\frac{f(\ldots) \to c, f(\ldots) \to d}{f(\ldots) \to c, c \approx d}$$

Collapse: $$\frac{f(\ldots, c, \ldots) \to c', c \to d}{f(\ldots, d, \ldots) \to c', c \to d}$$

Composition: $$\frac{f(\ldots) \to c, c \to d}{f(\ldots) \to d, c \to d}$$

Other Transition Rules

Extension:
$$\frac{C[g(\ldots)] \approx t}{C[c'] \approx t, g(\ldots) \to c'}$$

Simplification:
$$\frac{C[g(\ldots)] \approx t, g(\ldots) \to c'}{C[c'] \approx t, g(\ldots) \to c'}$$

Deletion:
$$\frac{s \approx s}{}$$

Orientation:
$$\frac{f(\ldots) \approx c}{f(\ldots) \to c}$$

Abstract Congruence Closure

A ground rewrite system $R = D \cup C$ is an *(abstract) congruence closure* (over Σ and $K \subset U$) *for* E if

1. Every normal form $c \in K$ represents a term $t \in \mathcal{T}(\Sigma)$ via R, and
2. R is a fully reduced, ground convergent rewrite system over terms in $\mathcal{T}(\Sigma \cup K)$.
3. For all terms $s, t \in \mathcal{T}(\Sigma)$, we have:

$$s \leftrightarrow_E^* t \text{ if and only if } s \downarrow_R t.$$

Example: Abstract Congruence Closure

Let

$$E_0 = \{f(a,b) \approx a, \;\; f(f(a,b),b) \approx b\}.$$

Then,

$$\begin{aligned}
D_0 &= \{a \to c_1,\ b \to c_2,\ f(c_1,c_2) \to c_1,\ f(c_1,c_2) \to c_2\}, \\
D_1 &= \{a \to c_1,\ b \to c_2,\ f(c_1,c_2) \to c_2\}, \;\; C_1 = \{c_1 \to c_2\}, \\
D_2 &= \{a \to c_2,\ b \to c_2,\ f(c_2,c_2) \to c_2\}.
\end{aligned}$$

The set D_2 is an abstract congruence closure for E_0.

What did we gain?

- Time complexity: Exponential to Polynomial jump
- Very simple ordering used
- Generalize to handle AC symbols: Suppose $\Sigma_{AC} \subset \Sigma$ defined to be AC. We additionally need completion procedure for commutative monoids. Almost straight-forward extension. Cf. regular ground AC-completion.
- Ground convergent systems:

$$\begin{array}{ccccc} E & \mapsto & R = D \cup C & \mapsto^? & E' \\ \Sigma & & \Sigma \cup K & & \Sigma(\text{ convergent}) \end{array}$$

- Non-symmetric relations: Non-termination to Polynomial jump

Complexity Analysis

1. Extension, Simplification, Orientation, Deletion Steps: $O(n)$ steps.
2. Superposition, Collapse, and Composition: $O(n\delta)$ steps, where δ is the depth of the ordering on K.

Consider a rule obtained after the first stage above:

$$\underline{f}\ (\underline{c_1},\ \underline{c_2}, \ldots,\ \underline{c_k})\ \rightarrow\ \underline{c}$$

Each marked position changes at most δ times and there are $O(n)$ such positions.

Time complexity: $O(n\delta)$.

Abstract Rewrite Closure

Reln	Ordered Inference Rule	Ground Case?	Our Method
$\approx$	superposition	EXP time	Poly time
$\rightarrow$	ordered partial para-modulation	non-terminating	Poly time
$\approx \cup \rightarrow$	combination	non-terminating	Poly time

Why?

Substructure sharing using new constants and extra flexibility in choosing ordering.

Combining Concepts from Different Areas

Interpretations for Extended Signatures:

The DAG interpretation for Congruence Closure Algorithms The constants $c_1, c_2, \ldots \in U$ are pointers (to vertices in a DAG) and a D-rule $f(c_1, c_2) \to c$ says that the c points to the DAG vertex with symbol f and pointers c_1 and c_2.

The states interpretation for Tree Automata The constants $c_1, c_2, \ldots \in U$ are states of an automaton and a D-rule $f(c_1, c_2) \to c$ represents a transition of a bottom-up tree automata.

In other words, we have combined techniques from tree automata, standard rewriting, and DAG-based implementations.

Deciding Confluence of Ground TRS

Consider

$$E_0 = \{a \to fab,\ fab \to fba\}$$

and

$$E_1 = \{gfa \to fgfa,\ gfa \to ffa,\ ffa \to fa\}$$

E_1 is confluent, i.e., any two congruent terms can be rewritten to a common term, e.g., $fga \leftrightarrow^*_{E_1} ffgfa$ and

$$fga \ \to^* \ fa \ \leftarrow^* \ ffgfa$$

whereas E_0 is not, e.g., $f(fba, b) \leftrightarrow^*_{E_0} fba$, but

$$f(fba, b) \ \to^* ?? \ \leftarrow^* \ f(b, a)$$

What was known?

- Reachability for GTRS is decidable in polynomial time: Using tree automata techniques
- Congruence for GTRS is decidable in polynomial time: Using dag based congruence closure algorithms
- Confluence for GTRS is decidable in exponential time: Using tree automata techniques (Ground Tree Transducers)
- Open: Polynomial time algorithm for deciding confluence of ground term rewrite systems?

Polynomial Time Algorithms

Let E be the input set of ground rewrite rules.

1. Construct an abstract rewrite closure $D \cup F \cup B$ for E
2. Construct an abstract congruence closure R for E (over the same extended signature used above)
3. Check that every pair of constants c and d in the same congruence class (in R) rewrite to some common term using $D \cup F \cup B$
4. Check that every constant c and signature $f(\ldots)$ in the same congruence class (in R) rewrite to some common term using $D \cup F \cup B$

References

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