Deciding Properties of RGV Systems

Using Rewrite Closures

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Rewriting Systems

Rewrite Systems: Rules for replacing terms by other terms

Useful for handling equality in automated deduction

Suppose: $E = \{c_1 \approx c_2, c_2 \approx c_3, c_4 \approx c_3\}$

Is c_3 reachable from c_1 ?

$$c_1 \quad \longrightarrow \quad c_2 \quad \longrightarrow \quad c_3$$

Does E imply $c_1 \approx c_4$?

$$\begin{array}{ccccccc} c_1 & \longrightarrow & c_2 & \longrightarrow & c_3 & & \\ & & & & c_3 & \longleftarrow & c_4 \end{array}$$

Properties of Rewriting Systems

Given a set R of rewriting rules:

1. Word Problem: Is $s \leftrightarrow_R^* t$?
2. Reachability: Is $s \rightarrow_R^* t$?
3. Joinability: $\exists u : \; s \rightarrow_R^* u \leftarrow_R^* t$?
4. Termination: $\exists s_1, s_2, \ldots : \; s_1 \rightarrow_R s_2 \rightarrow_R \cdots$?
5. Confluence: Is $\leftarrow_R^* \circ \rightarrow_R^*$ contained in $\rightarrow_R^* \circ \leftarrow_R^*$?
6. Unification Problem: $\exists \sigma : \; s\sigma \leftrightarrow_R^* t\sigma$?

Motivation: Specialized Systems

Theorem provers are best used in their incarnations as decision procedures for restricted fragments

Ground TRSs ***Rt Ground*** ***Collapsing*** ***General TR***

Expressiveness ⟶

Intractability ⟶

How far can you go using the specialized theorem proving approach?

Context: Special Classes

- Rewrite rules of the form: $c_1 \approx c_2$
  - Word problem: Union-Find data structure
  - Reachability: Existence of a directed path in a graph
  - Termination: Existence of a directed cycle in a graph
- Ground TRS: No variables in terms
  - Word problem: Congruence closure
  - Reachability: Ground tree transducers (tree automata)
  - Confluence: Tree automata [DHLT87:LICS, Oya87:TCS, CGN01:FOCS, Tiw01:LICS]

Context: Special Classes

- Shallow Linear TRS: All terms are shallow and linear
  - Word problem: Extended Congruence closure
  - Reachability: Rewrite Closure
  - Confluence: Decidable [GTV03:STACS]
- Right-ground TRS: No variables on the right-hand sides of rules
  - Word problem: Undecidable [Oya90:IEICE]
  - Reachability, Joinability: Decidable [OO93:IEICE]
  - Confluence: Decidable [GTV04:AAECC]

The Main Results

RGV Systems: All **r**ight-hand side terms are either **g**round or a **v**ariable

The following properties are decidable for these systems.

1. Reachability: Is $s \rightarrow^* t$?
2. Joinability: $\exists u : \; s \rightarrow^* u \leftarrow^* t$?
3. Termination: $\exists s_1, s_2, \ldots : \; s_1 \rightarrow s_2 \rightarrow \cdots$?
4. Confluence: Is $\leftarrow^* \circ \rightarrow^*$ contained in $\rightarrow^* \circ \leftarrow^*$?

The word problem for collapsing systems is undecidable.

Word problem and unification are decidable for confluent collapsing systems.

Significance of the Results

- Basic concepts that help in providing uniform description of decision procedures for various classes of TRSs
- The algorithms build upon those for simpler subcases:
  - Ground rewrite systems
  - Shallow linear rewrite systems
  - Right ground systems

Basic Concepts

The central concepts:

- Rewrite closure: transforms reachability proofs into certain kinds of valley proofs

  Constrained rewrite closure: joinability constraints

- Minimal conservative extensions: New constants in signature
- Confluence characterization using rewrite closures
- Top stable structures

Rewrite Closure

Transform reachability proofs $\rightarrow_R^*$ to valley proofs

R :	a rewrite system
(F, B):	rewrite closure for R: whenever $s \rightarrow_R^* t$, we have

$$\begin{tikzcd}
s \arrow[dr, "*", "F"'] & & t \\
& u \arrow[ur, "*", "B"'] &
\end{tikzcd}$$

- F not confluent, $\therefore$ non-deterministic search for u.
- Ground TRS: rewrite closure is a ground tree-transducer
- Rewrite closures can be constructed using a chaining calculus
- Which does not terminate in general

Rewrite Closure: Example

Let $R = \{0 \rightarrow -0,\ -x + x \rightarrow 0\}$

Variable chaining gives:

$$\begin{tikzcd} & --0+-0 \arrow[dr] & \\ -0+-0 \arrow[ur] & & 0 \end{tikzcd}$$

There is no valley proof for $-0 + -0 \rightarrow 0$ using R.

To get a rewrite closure, we need variable chainings

This may cause nontermination:

$$0+0 \rightarrow 0, \quad -0+-0 \rightarrow 0, \quad --0+--0 \rightarrow 0, \ldots$$

Constrained Rewrite Closures

Get finite rewrite closure presentations using joinability constraints.

Convert the nonlinear rule

$$f(x, y, gxy) \quad \rightarrow \quad y$$

to a linear constrained rule

$$f(x_1, y_1, gx_2y_2) \quad \rightarrow \quad y_3 \quad \{\{y_3; y_1, y_2\}, \{x_1, x_2\}\}$$

Meaning: y_3 is reachable from y_1 and y_2 and x_1 and x_2 are joinable

Example: Constrained Rewrite Closure

Let $R = \{0 \to -0, \ -x + x \to 0\}$

Constrained rewrite closure for R:

$$\begin{aligned} CF &= \{-x_1 + x_2 \to 0 \quad \{\{x_1, x_2\}\} \\ B &= \{0 \to -0\} \end{aligned}$$

Note that

$$- - 0 + - - 0 \to_{CF/B} 0$$

because -0 and $- - 0$ are B-joinable

But if RHS was a variable, then this rule may not be necessarily reducing

Minimum Joining Extension

When LHS variables are instantiated with constants, the RHS variable may not be.

$$f(x_1, y_1, gx_2y_2) \quad \rightarrow \quad y_3 \quad \{\{y_3; y_1, y_2\}, \{x_1, x_2\}\}$$

Introduce new constants to represent the *join of a finite subset of constants*

$$f(c_1, d_1, gc_2d_2) \quad \rightarrow \quad \underline{d_{\{1,2\}}} \quad \{\{d_1, d_2\}, \{c_1, c_2\}\}$$

Meaning: $Reach(\{d_{\{1,2\}}\}) = Reach(\{d_1, d_2\})$ whenever d_1 and d_2 are joinable

Rewrite Closure for RGV Systems

R: RGV system

(CF, B): constrained forward rules, $s \to \alpha$ **if** $\mathcal{C}$

flat backward rules, $c \to f c_1 \ldots c_k$

$(CF/B, B)$: rewrite closure for R ?

$(CF\sqrt{}B, B)$: rewrite closure for R

In the presence of the new constants,

if $s \to_R^* t$, then $s \to_{CF/B}^* \circ \to_B^* t$, and

$$\begin{tikzcd} s \arrow[dr, "*", "CF\sqrt{}B"'] & & t \\ & u \arrow[ur, "*", "B"'] & \end{tikzcd}$$

Consequences

Is $s \rightarrow_R^* t$?

- Find all u s.t. $s \rightarrow^*_{CF\sqrt{}B} u$
- Check if for some u, we have $u \rightarrow_B^* t$

Is $s \rightarrow_R^* \circ \leftarrow_R^* t$?

- Find all u_1, u_2 reachable from s, t via $CF\sqrt{}B$
- Check if u_1, u_2 are joinable by B

Is R terminating ?

- Check if there is a c s.t. $c \rightarrow_R^+ C[c]$

Characterizing Confluence

R	:	original rewrite system
(F, B)	:	rewrite closure for R w.r.t. $\succ$
B_{CC}	:	convergent presentation of B w.r.t. $\succ$

R is confluent iff

- $F \cup B_{CC}$ is locally confluent
- $\downarrow_{F \cup B_{CC}} \subseteq \downarrow_{F \cup B}$

In our case, F is actually $CF\sqrt{B}$

Adapting Confluence Characterization

CCF = canonical instances of the rules in CF

R is confluent iff

- $CCF \cup B_{CC}$ is locally confluent
- $\downarrow_{CCF \cup B_{CC}} \subseteq \downarrow_{CF\sqrt{B} \cup B}$

Consequently, the word problem and the unification problem for confluent R is decidable

Deciding Confluence

Additionally check that

$$\downarrow_{CCF \cup B_{CC}} \quad \subseteq \quad \downarrow_{CF\sqrt{B} \cup B}$$

- constants equivalent modulo $CCF \cup B_{CC}$ are B-joinable
- $TopStable(\alpha)$ has atmost one term
  $TopStable(\alpha) =$ signatures of terms equivalent to α that do not rewrite to a constant
- if $t = ft_1 \dots t_m \in TopStable(\alpha)$, then α is a constant, t is ground, and $\alpha \to_B^* fc_1 \dots c_m$ s.t. c_i and t_i are B_{CC}-equivalent.

Future Work

- More general sufficient conditions for guaranteeing that rewrite closures can be constructed:
  - RGV systems is one condition. Can it be generalized?
- Relationship with automata-theoretic methods
  - There is a close connection at the level of ground systems
- Can the chaining-based calculus (used for constructing rewrite closure) yield logical decision procedures for (fragments of) the theory of reals
  - The ordering $>$ relation is a nonsymmetric and (partially) monotone relation like $\rightarrow$

Conclusion

The following properties are decidable for RGV systems.

1. Reachability : rewrite closure
2. Joinability : rewrite closure
3. Termination : rewrite closure and RGV termination characterization
4. Confluence : confluence characterization, top stable structures

Word problem and unification are decidable for confluent collapsing systems.