Termination of Rewriting with Right-Flat Rules

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Abstract. Termination and innermost termination are shown to be decidable for rewrite systems whose right-hand side terms are restricted to be shallow (variables occur at depth at most one) and linear. Innermost termination is also shown to be decidable for shallow rewrite systems. In all cases, we show that nontermination implies nontermination starting from flat terms. The proof is completed by using the useful enabling result that, for right shallow rewrite systems, existence of nonterminating derivations starting from a given term is decidable. We also show that termination is undecidable for shallow rewrite systems. For right-shallow systems, general and innermost termination are both undecidable.

1 Introduction

Termination is an important property of computing systems and it has generated significant renewed interest in recent years. There has been progress on both the theoretical and practical aspects of proving termination of many different computing paradigms - such as term rewrite systems, functional programs, and imperative programs.

This paper focuses on termination of term rewrite systems. For results on termination of imperative programs, the reader is referred to [6, 11, 2, 1]. There has been steady progress on the problem of deciding if a term rewrite system is terminating. While termination is undecidable for general term rewrite systems and string rewrite systems [7], several subclasses with decidable termination problem have been identified. Termination is decidable in polynomial time for ground term rewrite systems [7, 9]. Termination is decidable for right-ground term rewrite systems [3] and also for the more general class that also has right-variable rules [4]. Later, termination was shown to be decidable for rewrite systems that contain right-ground, collapsing, or shallow right-linear rewrite rules [5]. There were further decidability results about shallow left-linear and shallow right-linear rewrite systems [12].

This paper presents a new decidability result that subsumes most of the above results. In particular, we show that termination (and innermost termination as well) is decidable for rewrite systems whose right-hand side terms are both shallow and linear, that is, variables occur at depth at most one and no variable occurs more than once on the right-hand side terms. There is no restriction on the left-hand side terms. Thus, right-ground systems and shallow right-linear systems are both contained in our class.

The attractiveness of our results comes partly from the remarkable simplicity of the proof. The proofs presented in this paper are self-contained and not dependent on any other results. At the top-level, we argue that if a right flat-linear rewrite system is nonterminating, then there is a nonterminating derivation starting from a constant or a ground flat term (Section 6). To complete the proof, we prove a (stronger) result: termination of right shallow systems *starting from a given term* is decidable (Section 4). In other words, nontermination is semi-decidable for right shallow systems. Using similar arguments, we also show that innermost termination of flat term rewrite systems is decidable (Section 5). This last result is in sharp contrast to the undecidability of general termination for flat rewrite systems (Section 7). We also show undecidability of general and innermost termination for right shallow rewrite systems (Section 7).

2 Preliminaries

A *signature* $\Sigma = \cup_i \Sigma_i$ is a finite set of function symbols and constants indexed by their *arity* i. Thus, Σ_0 is the set of constants in Σ. A *term* t over Σ is constructed from the symbols in Σ and a set of variables in the usual way. To reduce clutter, we write $ft_1 \ldots t_m$ instead of the standard notation $f(t_1, \ldots, t_m)$ for a term. A *position* is a string of natural numbers (including the empty string λ). We use . as the string concatenation operator and $|p|$ to denote the length of the string p. Given a term t and a position p, the subterm of t at position p, denoted by $t|_p$, is defined to be (i) t, if $p = \lambda$, (ii) $t_i|_{p'}$ if $t = ft_1, \ldots, t_m$ and $p = i.p'$ and $1 \leq i \leq m$, and (iii) $\perp$, otherwise. The set $\mathcal{P}os(t)$ consists of all positions p such that $t|_p \neq \perp$. The subterm $t|_p$ of t is said to be *at depth* $|p|$.

We write $p_1 > p_2$ (equivalently, $p_2 < p_1$) and say p_1 is below p_2 (equivalently, p_2 is above p_1) if p_2 is a proper prefix of p_1, that is, $p_1 = p_2.p_2'$ for some nonempty p_2'. Positions p and q are *disjoint* if $p \not\geq q$ and $q \not\geq p$. When p_1 is of the form $p_2.p_2'$, $p_1 - p_2$ denotes p_2'.

The *height* of a term t, $height(t)$, is zero if t is a variable or a constant, and it is $1 + max\{height(t_i) : i = 1, \ldots, m\}$ if $t = ft_1 \ldots t_m$. A term t is said to be *flat* if its height is at most one. A term t is *shallow* if all variables in t occur at depth at most one. A term t is *linear* if no variable occurs more than once in t.

The set of all subterms of a term s is denoted by $\lfloor s \rfloor$. A term t is said to be *reachable* from s if $s \rightarrow_R^* t$. A term t is said to be *context-reachable* from s if there exists a context $C[\_]$ such that $s \rightarrow_R^* C[t]$.

A *rewrite system* R is a finite set of rewrite rules $l \rightarrow r$, where l and r are terms. A *rewrite step*, using the rule $l \rightarrow r$, applied to a term s at position p, is denoted by $s \rightarrow_{l \rightarrow r, \sigma, p} s[r\sigma]_p$, where σ is a substitution such that $l\sigma \equiv s|_p$. We do the usual assumptions for the rules $l \rightarrow r$ of a rewrite system R, i.e. l is not a variable, and all variables occurring in r also occur in l. A rewrite system R is *terminating from* s if there are no infinite R*-derivations*, $s \to_R s_1 \to_R \cdots$. If R is terminating from every term, then R is said to be *terminating*.

A term s is R*-irreducible* if there is no term t such that $s \to_R t$. A rewrite step $s \to_{R,p} t$ is an *innermost rewrite step* if $s|_{p'}$ is R-irreducible, for all $p' > p$. The concepts of reachability and termination can be naturally defined for innermost rewriting.

A rewrite system R is (right-)shallow, respectively flat/linear, if all (right-hand side) terms in R are shallow, respectively flat/linear.

3 Flattening and Other Simplifying Assumptions

The discussion of this section is written for general termination, but it is also valid when we interpret termination as innermost termination, reachability as innermost reachability, and so on. To this end, in the innermost case we assume that for a given TRS, all the rules $l \to r$ such that l has a proper subterm that is not a normal form have been removed. Note that these rules can not be used in an innermost derivation.

For purposes of deciding termination of R, we show that we can assume, without loss of any generality, that all shallow terms in the rewrite system R are indeed flat. This observation follows from the following result.

Proposition 1. *Let R be a rewrite system containing a rewrite rule $l[u] \to r[v]$ such that u and v are ground. Let c and d be two new constants. Then, R is (innermost-)terminating iff $R' := R - \{l \to r\} \cup \{l[c] \to r[d], u \to c, d \to v\}$ is (innermost-)terminating.*

Proof. (Sketch) Any rewrite step $s \to_R t$ can be simulated by a R'-derivation: if the R rewrite step does not use $l \to r$, then $s \to_{R'} t$; if it does, then $s := s[l[u]]$ and $t := s[r[v]]$ and the corresponding R'-derivation is $s[l[u]] \to s[l[c]] \to s[r[d]] \to s[r[v]]$.

Any R'-derivation $s_1 \to_{R'} s_2 \to_{R'} \cdots$ can be simulated by the R-derivation $s_1\sigma \to_R^* s_2\sigma \to_R^* \cdots$, where σ is the substitution $\{c \mapsto u, d \mapsto v\}$. It can be seen that infinite R'-derivations necessarily map to infinite R-derivations. □

As a second simplifying assumption, we note that we can assume, again without loss of generality, that the signature Σ of R contains constants and exactly one m-ary function symbol. If Σ contains multiple n-ary function symbols ($n > 0$), then we choose m to be one plus the maximum arity of a function symbol in Σ. Let $\Sigma' := \Sigma'_0 \cup \Sigma'_m$, where $\Sigma'_0 := \Sigma$ and $\Sigma'_m = \{f\}$. Now, if $g(s_1, \ldots, s_n)$ is a term over Σ, we encode it by the term $f(\mathit{encode}(s_1), \ldots, \mathit{encode}(s_n), g, \ldots, g)$ over Σ', where $\mathit{encode}(s_i)$ recursively encodes s_i and the $g \in \Sigma'_0$ is used as padding to get m arguments for f. This encoding of terms can be extended to rewrite systems. It is easy to see that this encoding preserves the termination property of rewrite systems.

4 Right Flat Systems

In this section, we will show that, given a right shallow rewrite system R and a term s, it is decidable if R is terminating from s. In particular, this implies that nontermination is semi-decidable for right shallow rewrite systems. We will show that termination is undecidable for right shallow systems in Section 7. A right shallow system can be transformed into a right flat rewrite system, while preserving its termination property, similarly to what is shown in Section 3. Henceforth, in this section, R is a right flat TRS.

The proofs of this section are written for general termination, but they are also valid when we interpret termination as innermost termination, reachability as innermost reachability, and so on.

An important property of a right flat system R is that if $s \to_R^* t$, then every subterm of t is reachable from either a constant or some subterm of s. We prove this by inductively marking each position of a term (in the above derivation) by the witness term from $\lfloor s \rfloor \cup \Sigma_0$ as follows:
Base Case: For the term s, the marking is given by $M_s(p) = s|_p$ for all positions p of s.
Induction Step: Consider the rewrite step, $u \to_{l\to r,p} v$, where we have the mapping $M_u : \mathcal{P}os(u) \mapsto \lfloor s \rfloor \cup \Sigma_0$. We construct the mapping $M_v : \mathcal{P}os(v) \mapsto \lfloor s \rfloor \cup \Sigma_0$ as follows:

1. if $p' \not> p$, then $M_v(p') := M_u(p')$,
2. if $p' = p.p_0$, $|p_0| = 1$, and $r|_{p_0}$ is a constant, then $M_v(p') := r|_{p_0}$.
3. for every p_0 such that $r|_{p_0}$ is a variable we choose any position q_0 in l such that $l|_{q_0} = r|_{p_0}$ and then for every $p' = p.p_0.p_1$, $|p_0.p_1| \geq 1$, we define $M_v(p') := M_u(p.q_0.p_1)$.

Note that we are assuming that every variable on the right-hand side also appears on the left-hand side; if not, then the rewrite system is trivially nonterminating. One easy property of the markings is that $v|_p$ is reachable from $M_v(p)$.

Lemma 1. *For a right flat R, if the derivation $s \to_R^* t$ is marked as above, then $t|_p$ is (innermost-)reachable from $M_t(p)$ for all $p \in \mathcal{P}os(t)$. In particular, if s is a constant, all subterms of t are (innermost-)reachable from a constant.*

Moreover, $M_t(\lambda) = s$, and if s is not a constant, then for any $p \in \mathcal{P}os(t)$ with $p \neq \lambda$ we have $M_t(p) \neq s$.

Proof. The claim is proved inductively on the length of the derivation $s \to_R^* t$. The base case is trivial. For the induction step, suppose $u \to_{l\to r,p} v$ and $u|_{p'}$ is reachable from $M_u(p')$ for all $p' \in \mathcal{P}os(u)$. Consider any $p' \in \mathcal{P}os(v)$. We show that $v|_{p'}$ is reachable from $M_v(p')$ as follows:

- If $p' \leq p$, then we have $M_v(p') = M_u(p') \to_R^* u|_{p'} \to_{l\to r,p-p'} v|_{p'}$.
- If $p' = p.p_0$, $|p_0| = 1$, and $r|_{p_0}$ is a constant, then $M_v(p') = r|_{p_0}$ is reachable from $v|_{p'} = r|_{p_0}$.
- If $p' = p.p_0.p_1$, $|p_0.p_1| \geq 1$, and $r|_{p_0}$ is a variable, then, for some q_0, $M_v(p') = M_u(p.q_0.p_1)$, $v|_{p'} = u|_{p.q_0.p_1}$, and $u|_{p.q_0.p_1}$ is reachable from $M_u(p.q_0.p_1)$ by induction hypothesis.

– If $p' \parallel p$, then the claim holds by induction hypothesis again as $v|_{p'} = u|_{p'}$ and $M_v(p') = M_u(p')$.

The second statement of the lemma follows from the construction of the mapping function. This completes the proof. □

A second property of markings is that $M_v(p)$ is context-reachable from $M_v(p')$ for all $p' \leq p$.

Lemma 2. *For a right flat R, if the (innermost-)derivation $s \to_R^* t$ is marked as above, then for all $p, p' \in \mathcal{P}os(t)$ such that $p' < p$, $M_t(p)$ is (innermost-)context reachable from $M_t(p')$. Moreover, if $M_t(p)$ and $M_t(p')$ are both constants, then $M_t(p)$ is (innermost-)context reachable from $M_t(p')$ in one or more steps.*

Proof. The claim is proved inductively on the length of the derivation $s \to_R^* t$. The base case is trivial. For the induction step, suppose $u \to_{l \to r, p} v$ is the last step of the derivation and the claim is true for u. Consider any $p' \in \mathcal{P}os(v)$. We prove the claim for all $p'' < p'$ as follows:

– If $p' \leq p$ or $p' \parallel p$, then we have $M_v(p') = M_u(p')$ and $M_v(p'') = M_u(p'')$ for all $p'' \leq p'$, and hence, by induction hypothesis, the claims follow.
– If $p' = p.p_0$, $|p_0| = 1$, and $r|_{p_0}$ is a constant, then $M_v(p') = r|_{p_0}$. Now consider any $p'' < p'$. By Lemma 1, $v|_{p''}$ is reachable from $M_v(p'')$. Note that $M_v(p')$ is a subterm of $v|_{p''}$. Hence, $M_v(p')$ is context reachable from $M_v(p'')$.
  If $M_v(p'')$ is a constant, then $v|_{p''}$ is reachable from $M_v(p'')$ *in one or more steps*. Hence, $M_v(p')$ is context reachable from $M_v(p'')$ in one or more steps.
– If $p' = p.p_0.p_1$, $|p_0.p_1| \geq 1$ and $r|_{p_0}$ is a variable, then, for some q_0, $M_v(p') = M_u(p.q_0.p_1)$. There are two cases. (a) If $p'' \leq p$, then $M_v(p'') = M_u(p'')$ and, by induction hypothesis, $M_u(p.q_0.p_1)$ is context reachable from $M_u(p'')$, which is the same as saying that $M_v(p')$ is context reachable from $M_v(p'')$. (b) If $p'' > p$, then $M_v(p'') = M_u(p.q_0.p_1')$ for some $p_1' < p_1$ and, by induction hypothesis, $M_u(p.q_0.p_1)$ is context reachable from $M_u(p.q_0.p_1')$. This is the same as saying that $M_v(p')$ is context reachable from $M_v(p'')$. In both cases, the second part of the claim is immediate by induction.

This completes the proof. □

Finally, a third observation about the markings is that positions below $height(s)$ are always marked by constants.

Lemma 3. *For a right flat R, if the (innermost-)derivation $s \to_R^* t$ is marked as above, then for all $p \in \mathcal{P}os(t)$ such that $|p| > height(s)$, $M_t(p)$ is always a constant.*

Proof. This is proved inductively on the length of the derivation. For the base case, $t = s$ and the claim is vacuously true. We just need to show that the property is preserved by every rewrite step, say $u \to_{l \to r, p} v$. Consider any position $p' \in \mathcal{P}os(v)$ such that $|p'| > height(s)$.

(i) If $p' \parallel p$ or $p' \leq p$, then $M_v(p') = M_u(p')$ and by induction hypothesis $M_u(p')$ will be a constant.
(ii) If $p' = p.p_0$, $|p_0| = 1$ and $r|_{p_0}$ is a constant, then $M_v(p') = r|_{p_0}$, which is a constant.
(iii) If $p' = p.p_0.p_1$, $|p_0.p_1| \geq 1$ and $r|_{p_0}$ is a variable, then $M_v(p') = M_u(p.q_0.p_1)$ for some q_0 and $|p.q_0.p_1| \geq |p'|$. Hence induction hypothesis is applicable and we can conclude that $M_u(p.q_0.p_1)$, and therefore $M_v(p')$, is a constant. □

A simple consequence of Lemma 2 and Lemma 3 is that if R is terminating from s, then the height of terms reachable from s is bounded.

Corollary 1. *Let R be a right flat TRS, s any term. Then, if R is (innermost) terminating from s, then for any term t (innermost) reachable from s, we have $height(t) \leq height(s) + |\Sigma_0|$.*

Proof. We mark the derivation $s \to_R^* t$ as above. Now, suppose $height(t) > height(s) + |\Sigma_0|$. By Lemma 3, each position in t that is deeper than $height(s)$ is marked with a constant. Since $height(t) > height(s) + |\Sigma_0|$, by pigeon-hole principle, there are two positions $p, p' \in \mathcal{P}os(t)$ such that $p < p'$ and $M_t(p) = M_t(p')$ and $M_t(p)$ is a constant, say c. By Lemma 2, it follows that c is context reachable from c in one or more steps. Moreover, by Lemma 1 the position λ of every term in a derivation is marked with s. Using Lemma 2 again, we infer that $M_t(p)$, or c, is also context reachable from s. Thus, we have a derivation $s \to_R^* C_1[c] \to_R^+ C_1[C_2[c]] \to_R^+ C_1[C_2[C_2[c]]] \to_R^+ \ldots$. Hence, there is an infinite derivation starting from s. □

Using the above corollary, we can show that the existence of nonterminating derivations starting from a term is decidable for right flat systems.

Theorem 1. *Termination (innermost-termination) of a right flat TRS R from a given term is decidable. Hence, nontermination (innermost-nontermination) is semi-decidable for right flat rewrite systems.*

Proof. Let s be any term. We enumerate all derivations starting from s. If we reach a term with height greater than $height(s) + |\Sigma_0|$, then by corollary 1 we know that R is nonterminating from s. Otherwise, we will get only *finitely* many reachable terms. If there is a derivation that cycles among these terms, then R is nonterminating from s. If not, then R is terminating from s. □

Remark: We state below a related theorem that uses a similar argument as the proof of Theorem 1.

Theorem 2. *Let $\mathcal{C}$ be a class of TRS's that are effectively regularity preserving, and s a term. Termination of the TRS's of $\mathcal{C}$ from s is decidable.*

Proof. (sketch) The regular set S of terms reachable from s can be checked to be infinite, in which case we know that there exists an infinite derivation from s. Otherwise, we have a finite number of terms reachable from s, and we can check for the existence of a cycle. □

We shall not use Theorem 2 in this paper. However, we note here that, using recent results on regularity preserving systems [10], we immediately get very simple proofs of known decidability results, such as for right-ground systems [3]: a right-ground system is regularity preserving, and is non-terminating iff it is non-terminating from some right-hand side.

5 Innermost Termination of Flat TRS's

In this section, we show that innermost termination of flat rewrite systems is decidable. In sharp contrast, general termination is undecidable for flat rewrite systems (Section 7).

Let R be a flat rewrite system. We show decidability of innermost termination of flat systems by showing that for nonterminating R, there is an infinite derivation starting with either a constant or a flat term. Using Theorem 1, we know that these checks are decidable.

Lemma 4. *Let R be a flat rewrite system that is not innermost terminating. Then, there is an infinite innermost derivation starting from a constant or a ground flat term.*

Proof. We assume that there is no infinite derivation from a constant and we show that there is one from a flat term.

Since R is not innermost terminating, there exists an infinite innermost derivation $t_0 \to_R t_1 \to_R \dots$ whose first step is at position λ. We first prove that for every i, *every subterm at depth* 1 *of t_i is either reachable from a constant, or in normal form.* First note that no term t_i is a constant, by our initial assumption. Now, we finish the proof of the claim by marking the derivation as above. By construction of marking, for any i, the mark at any depth 1 subterm of t_i is either a constant or a *proper* subterm of t_0. Now, since we use innermost rewriting, all proper subterms of t_0 are in normal form. By Lemma 1, all subterms at depth 1 of t_i are reachable from their markings, hence they are reachable from constants, or they are in normal form.

Now, we note that there exists at least one constant, call it c, that is in normal form. If not, any ground term can be innermost rewritten to another ground term, and hence there will be infinite derivations from constants, which contradicts our initial assumption.

We construct a new derivation $t'_0 \to_{R,\lambda} t'_1 \to_R \dots$ by defining each t'_i to be as t_i but replacing every subterm at depth 1 that is not reachable from any constant by the constant c chosen above. We need to show that the new derivation is "correct", that is, there is a rewrite step from t'_{i-1} to t'_i. Consider the corresponding step $t_{i-1} \to_{l \to r, p} t_i$.
(i) If $p \neq \lambda$, then p occurs inside a subterm at depth 1 that is necessarily reachable from a constant, since otherwise this subterm would be in normal form. Therefore, the same rewrite step can be applied on t'_{i-1} to produce t'_i.
(ii) If $p = \lambda$, then, by our initial assumption, both l and r are not constants. Moreover, r can not be a variable, since, otherwise, t_i would be in normal form

(and the derivation would be finite). Hence, $l \to r$ is of the form $f\alpha_1 \dots \alpha_m \to f\beta_1 \dots \beta_m$. If σ is the substitution used in this rewrite step, let σ' be as σ except for the cases where $x\sigma$ is not reachable from a constant, in which case we define $x\sigma' = c$. Clearly, $t'_{i-1} \to_{l\to r,\sigma',\lambda} t'_i$.

The infinite derivation $t'_0 \to t'_1 \to \dots$ is again innermost, and its initial term t'_0 satisfies that all their subterms at depth 1 are reachable from constants. Therefore, there exists a flat term s such that $s \to^*_R t'_0$, and hence, there exists an infinite derivation from a flat term s. $\square$

Lemma 4 and Theorem 1 together imply the following result.

Theorem 3. *Innermost termination is decidable for flat TRS's.*

Proof. Since there are only finitely many constants and ground flat terms, using Theorem 1, we can check if a given flat TRS R is not innermost terminating starting from one of these terms. By Lemma 4, we will find a witness for non-termination this way iff R is not innermost terminating.

6 Termination and Innermost Termination of Right Flat-Linear TRS's

A *right flat-linear* rewrite system has terms that are both flat and linear as right-hand side terms in all rewrite rules. In this section, we show decidability of termination and innermost termination for right flat-linear systems. Again, the proofs of this section are written for general rewriting, but they remain valid for innermost rewriting.

The proof of decidability of (innermost) termination for right flat-linear systems depends on two key observations. The first one is Lemma 1, which says that for any (innermost) derivation $s \to^*_R t$ using a right flat R, every *proper* subterm of t is (innermost) reachable from either a constant or a *proper* subterm of s.

The second key lemma is stated by first defining the following *measure* of a term t:

$$|t| := |\{p \mid t|_p \text{ is not (innermost) reachable from a constant } \}|$$

Note that $|t|$ depends on whether we are dealing with general or innermost rewriting. The next lemma uses right linearity of R as well.

Lemma 5. *Let R be a right flat-linear TRS. If s (innermost) rewrites to t, then $|s| \geq |t|$. Moreover, if $s[fs_1 \dots s_m]_p$ (innermost) rewrites to t at position p with a rule $fl_1 \dots l_m \to r$, and $|s| = |t|$, then, for every i in $\{1 \dots m\}$, if s_i is not (innermost) reachable from a constant, then l_i is a variable.*

Proof. Let $s \to_{l\to r,p} t$ be the rewrite step of the lemma. We prove the first statement by constructing an injective map from positions p' of t such that $t|_{p'}$ is not reachable from a constant to positions p'' of s such that $s|_{p''}$ is not reachable from a constant, as follows. If $p' \parallel p$ or $p' \leq p$, then we make $p'' := p'$. If $p' > p$, then p' can be written of the form $p.p_0.p_1$ where $r|_{p_0}$ is a height 0 term. In fact, $r|_{p_0}$ can not be a constant since otherwise $t|_{p'}$ would be a constant, and hence, it is a variable. We choose a position p'_0 such that $l|_{p'_0}$ is the same variable and set $p'' := p.p'_0.p_1$. The injectivity of the map follows by right linearity of R. Hence, $|s| \geq |t|$.

For the second statement, we are given that $|s| = |t|$, s is of the form $s[fs_1 \ldots s_m]_p$ and l is of the form $fl_1 \ldots l_m$. If a certain s_i is not reachable from a constant, but l_i is not a variable, then $p.i$ is not in the image of the previous mapping, and hence $|s| > |t|$, contradicting $|s| = |t|$. Therefore, all such l_i's are variables. □

The idea of the decidability proof is the same as that for Theorem 3, that is, we show that if R is non-terminating, then it is non-terminating from a constant or a flat term.

Lemma 6. *Let R be a right flat and right linear rewrite system. If R is nonterminating (innermost-nonterminating), then there exists an infinite (innermost-) derivation from a constant or a flat ground term.*

Proof. Assume that there is no infinite derivation from a constant. We will show that there is one from a flat term.

Since R is non-terminating, there exists an infinite derivation $t_0 \to_R t_1 \to_R \ldots$ that we choose to be the one with minimal height for t_0. Note that no term t_i is a constant, by our initial assumption, and hence, if a rule of the form $l \to r$ is applied at λ, then neither l nor r is a constant.

First we show that *no collapsing rule is applied at λ*. This is immediate from Lemma 1: if $t_{i-1} \to_R t_i$ is the first collapsing rule applied at λ, then by Lemma 1, all subterms at depth 1 of t_{i-1} are either reachable from a constant or a proper subterm of t_0. Since t_i is a proper subterm of t_{i-1}, it is reachable from either a constant or a proper subterm of t_0. In either case we have an infinite derivation starting from a term with smaller height than t_0, which contradicts our choice of t_0.

Now, we show that *there are infinite rewrite steps at position λ*. Suppose not. Let $t_{i-1} \to_R t_i$ be the last rewrite step at position λ. Then, there is an infinite derivation starting from some subterm at depth 1 of t_i. As before, this subterm is reachable from either a constant or a proper subterm of t_0. Again, we have an infinite derivation that contradicts the minimality of t_0.

Hence, the infinite derivation $t_0 \to_R \ldots$ has infinite rewrite steps at position λ, and all those steps are using rules of the form $l \to f\alpha_1 \ldots \alpha_m$, where the height of l is greater than or equal to 1. By Lemma 5, $|t_{i-1}| \geq |t_i|$ for all i. Since this relation can not be indefinitely decreasing, for some n we have $|t_n| = |t_{n+1}| = |t_{n+2}| = \ldots$. From the infinite derivation $t_n \to_R t_{n+1} \to_R \ldots$ we construct a new infinite derivation $t'_n \to_R^{0,1} t'_{n+1} \to_R^{0,1} \ldots$ as follows. Analogously to the proof of Lemma 4, we can deduce that there exists at least one constant c that is a normal form (this is true for the innermost and the general case). For every t_i, we construct t'_i to be equal to t_i except for the subterms at depth

1 that are not reachable from constants, which are replaced by c. Formally, $t'_i = t_i[c]_{j_1} \dots [c]_{j_k}$ if $t_i|_{j_1}, \dots, t_i|_{j_k}$ are the subterms at depth 1 in t_i that are not reachable from constants.

We show that the new derivation is correct by analyzing each rewrite step $t_{i-1} \to_R t_i$ and its corresponding $t'_{i-1} \to_R^{0,1} t'_i$.
(i) If $t_{i-1} \to_R t_i$ is at a position inside a subterm at depth 1 of t_{i-1} that is reachable from a constant, then, the same rewrite step can be applied on t'_{i-1} to produce t'_i.
(ii) If $t_{i-1} \to_R t_i$ is at a position inside a subterm, say $t_{i-1}|_j$, at depth 1 of t_{i-1} that is not reachable from a constant, then, $t'_{i-1} = t'_i$. This is because $t_i|_j$ is also not reachable from a constant, as $|t_i|_j| = |t_{i-1}|_j| \geq 1$, and by Lemma 1, if $t_{i-1}|_j$ was reachable from a constant then all its subterms would be, and $|t_{i-1}|_j| = 0$. Hence, $t'_{i-1} \to^0 t'_i$.
(iii) If $t_{i-1} \to_R t_i$ is at position λ, then, by Lemma 5, if $fl_1 \dots l_m \to r$ and σ are the rule and substitution applied, then l_k is a variable for every position k such that $t_{i-1}|_k$ is not reachable from a constant. We define a new substitution σ' to be equal to σ except for such variables l_k, for which we define $l_k\sigma' = c$. The same rule $fl_1 \dots l_m \to r$ applied to t'_{i-1} at position λ and with substitution σ' produces t'_i.

Since every rewrite step $t_{i-1} \to_R t_i$ at position λ corresponds to a rewrite step $t'_{i-1} \to_R^1 t'_i$, and there are infinite such steps, it follows that the derivation $t'_n \to_R^{0,1} t'_{n+1} \to_R^{0,1} \dots$ is infinite. Moreover, all subterms at depth 1 in t'_n are reachable from constants. Therefore, there exists a flat term t such that $t \to_R^* t'_n$, and hence, there exists an infinite derivation from a flat term t. □

Now, the main result follows immediately from Lemma 6 and Theorem 1.

Theorem 4. *Termination and innermost termination are both decidable for right shallow-linear rewrite systems.*

7 Undecidability Results

In this section, we prove undecidability of termination for shallow systems. This is done by a reduction from the Post correspondence problem (PCP) restricted to nonempty strings. Since nontermination is semi-decidable for shallow rewrite systems (Theorem 1), we will reduce the restricted PCP to nontermination of shallow rewrite systems.

Theorem 5. *Termination of shallow rewrite systems is undecidable.*

Proof. Consider an instance $\langle u_1, v_1\rangle \dots \langle u_n, v_n\rangle$ of the restricted PCP problem, that is, u_i, v_i are nonempty strings over alphabet Σ. We construct a shallow rewrite system R such that this PCP instance has a solution iff R is non-terminating.

Let $L = \mathtt{Max}(|u_1|, \ldots, |u_n|, |v_1|, \ldots, |v_n|) + 2$. We construct R over a signature Σ' given by

$$\begin{aligned}
\Sigma' &:= \Sigma'_0 \cup \Sigma'_1 \cup \Sigma'_2 \cup \Sigma'_6 \cup \Sigma'_8 \\
\Sigma'_1 &:= \Sigma \cup \{U_{i,j}, V_{i,j}, P_{i,j} : i = 1 \ldots n, j = 1 \ldots L\}, \\
\Sigma'_0 &:= \{U, U', V, V', P, P', P'', A, A', A''\} \\
\Sigma'_2 &:= \{f_1\}, \quad \Sigma'_6 \; := \; \{f_3\}, \quad \Sigma'_8 \; := \; \{f_2\}
\end{aligned}$$

The j'th symbol of u_i and v_i, whenever it exists, is denoted by $u_{i,j}$ and $v_{i,j}$ respectively. The rewrite system R is defined as follows:

$$\begin{aligned}
R &:= R_U \cup R_V \cup R_{2P} \cup R_{P'} \cup R_{P''} \cup R_Q \cup R_\alpha \cup R_w \cup R_f \\
R_U &:= \{U_{i,1}U_{i,2} \cdots U_{i,L}(U) \to U',\ U_{i,1}U_{i,2} \cdots U_{i,L}(U') \to U' : i = 1, \ldots, n\} \\
R_V &:= \{V_{i,1}V_{i,2} \cdots V_{i,L}(V) \to V',\ V_{i,1}V_{i,2} \cdots V_{i,L}(V') \to V' : i = 1, \ldots, n\} \\
R_{2P} &:= \{U_{i,j}(x) \to P_{i,j}(x), V_{i,j}(x) \to P_{i,j}(x) : i = 1, \ldots, n, j = 1, \ldots, L\} \\
R_{P'} &:= \{P_{i,1}P_{i,2} \cdots P_{i,L}(P') \to P' : i = 1, \ldots, n\} \\
R_{P''} &:= \{P_{i,1}P_{i,2} \cdots P_{i,L}(P'') \to P'' : i = 1, \ldots, n\} \\
R_Q &:= \{U \to P, U \to A, V \to P, V \to A, A \to A', A \to A'', P \to P', P \to P''\} \\
R_\alpha &:= \{\alpha(A') \to A', \alpha(A'') \to A'' : \alpha \in \Sigma\} \\
R_w &:= \{U_{i,j}(x) \to u_{i,j}(x) : 1 \le j \le |u_i|\} \cup \{U_{i,j}(x) \to x : j > |u_i|\} \\
&\quad \cup \{V_{i,j}(x) \to v_{i,j}(x) : 1 \le j \le |v_i|\} \cup \{V_{i,j}(x) \to x : j > |v_i|\} \\
R_f &:= \{f_1(x,y) \to f_2(x,y,x,y,x,y,x,y), f_2(x,y,z,z,t,t,U',V') \to f_3(x,y,z,z,t,t), \\
&\quad f_3(x,y,A',A'',P',P'') \to f_1(x,y)\}
\end{aligned}$$

$\Rightarrow$: We first show that if the PCP instance has a solution, then R is non-terminating. Let $w = u_{i_1} \ldots u_{i_k} = v_{i_1} \ldots v_{i_k}$ be a solution of the PCP instance. We show the existence of an infinite R-derivation starting from the ground term $s_1 := f_1(s_{uu}, s_{vv})$, where

$$\begin{aligned}
s_{uu} &:= U_{i_1,1} \ldots U_{i_1,L} \ldots U_{i_k,1} \ldots U_{i_k,L}(U) \\
s_{vv} &:= V_{i_1,1} \ldots V_{i_1,L} \ldots V_{i_k,1} \ldots V_{i_k,L}(V)
\end{aligned}$$

We rewrite s_1 using R_f to get the term $s_2 := f_2(s_{uu}, s_{vv}, s_{uu}, s_{vv}, s_{uu}, s_{vv}, s_{uu}, s_{vv})$. Now, we do not touch positions 1 and 2 of s_2. We rewrite position 3 using $R_w \cup \{U \to A\}$ to $w(A)$. We rewrite position 4 using $R_w \cup \{V \to A\}$ to $w(A)$. We rewrite position 5 with $R_{2P} \cup \{U \to P\}$ to $s_{pp} := P_{i_1,1} \ldots P_{i_1,L} \ldots P_{i_k,1} \ldots P_{i_k,L}(P)$. Similarly, we rewrite position 6 with $R_{2P} \cup \{V \to P\}$ to s_{pp}. We rewrite positions 7 and 8 with R_U and R_V respectively to U' and V'. As a result, we reach the term $s_3 := f_2(s_{uu}, s_{vv}, w(A), w(A), s_{pp}, s_{pp}, U', V')$.

Now, we use R_f to rewrite s_3 to $s_4 := f_3(s_{uu}, s_{vv}, w(A), w(A), s_{pp}, s_{pp})$. Now we use $R_\alpha \cup \{A \to A'\}$ in position 3 to get A'. Similarly, we rewrite position 4 to A''. In position 5, we use $R_{P'} \cup \{P \to P'\}$ to get P'. Similarly, we rewrite position 6 to P''. This way we get the term $s_5 := f_3(s_{uu}, s_{vv}, A', A'', P', P'')$, which is rewritten by R_f to the starting term $s_1 := f_1(s_{uu}, s_{vv})$.

$\Leftarrow$: Suppose R does not terminate. We need to show that the PCP instance has a solution. First, note that since all rules in R are height-preserving or decreasing, there is an infinite derivation with infinite rewrite steps at the top. We pick such an infinite derivation, but with minimal height initial term t. Then, the head of t has to be one of the f_i's. Otherwise, no infinite steps can be done at the top preserving the height. Therefore, we have an infinite sequence $f_1(\ldots) \to^* f_2(\ldots) \to^* f_3(\ldots) \to^* f_1(\ldots) \to^* \ldots$ at the top. We can assume that we start with a term of the form $f_1(u, v)$. By observing the R_f rules, one can deduce that u and v reach A', A'', P', P'', and that u reaches U' and that v reaches V'. This is possible only if the terms u and v are of the form

$$s_{uu} := U_{i_1,1} \ldots U_{i_1,L} \ldots U_{i_k,1} \ldots U_{i_k,L}(U)$$
$$s_{vv} := V_{j_1,1} \ldots V_{j_1,L} \ldots V_{j_{k'},1} \ldots V_{j_{k'},L}(V)$$

But, moreover, these terms have to be joinable to a term of the form $P_{i_1,1} \ldots P_{i_k,L}$ (P), and also of the form $P_{j_1,1} \ldots P_{j_{k'},L}(P)$. (Note here that since u_i, v_i are not λ, $U_{i,1} \ldots U_{i,L}(x)$ can not rewrite to x and hence the indices $i_1, \ldots, i_k, j_1, \ldots, j_{k'}$ will be preserved in any joinability proof.) Hence, $k = k'$ and $i_r = j_r$ for all r. But moreover, u and v have to be joinable to a term of the form $u_{i_1,1} \ldots u_{i_k,L}(A) = v_{i_1,1} \ldots v_{i_k,L}(A)$. Hence, $u_{i_1} \ldots u_{i_k} = v_{i_1} \ldots v_{i_k}$ and there is a solution of the original PCP problem. □

Theorem 6. *Termination and innermost termination of right-shallow rewrite systems is undecidable.*

Proof. Given an instance $\{(u_i, v_i) : i = 1, \ldots, n\}$ of Post correspondence problem, we generate the rewrite system $R = \{f(x) \to g(x, x, x),\ g(x, u_i(y), v_i(z)) \to h(x, y, z),\ h(x, u_i(y), v_i(z)) \to h(x, y, z),\ h(x, \epsilon, \epsilon) \to f(x)\}$. Here ϵ is a constant representing the empty string. Note that R is right-shallow. It is easy to see that the PCP instance has a solution iff R is (innermost) non-terminating.

We remark here that the recently published proofs of undecidability of reachability, joinability and confluence of flat systems also uses a reduction from PCP [8]. However, there are important differences and the construction in [8] can not be used directly in the proof of Theorem 5.

8 Conclusions

In this paper we showed that, given a right shallow rewrite system and a term, the existence of a (innermost) nonterminating derivation starting from the given term is decidable. Using this general result, we showed that innermost termination is decidable for shallow rewrite systems. We also used the same result to show decidability of general and innermost termination for right shallow-linear rewrite systems. We demonstrated that dropping assumptions on the rewrite systems leads to undecidability. In particular, general termination for shallow rewrite systems is undecidable and both general and innermost termination for right shallow rewrite systems is undecidable. As a result, we have considerably narrowed the gap between decidability and undecidability of termination of rewrite systems.

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