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BOX INVARIANCE OF HYBRID AND SWITCHED SYSTEMS

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Abstract: This paper investigates the concept of box invariance for classes of hybrid and switched systems. After motivating and defining the notion, we present a concise summary of results on its characterization for single-domain dynamical systems. The notion is then extended to the case of hybrid and switched systems. We provide sufficient conditions for a hybrid or switched system to be box invariant. Models of many real systems, especially those drawn from biology, have been found to be box invariant. This paper illustrates the concept using a pharmacodynamic model of blood glucose metabolism.

Keywords: Invariant Sets, Hybrid and Switched Dynamical Systems.

1. INTRODUCTION

The concept of box invariance has been recently introduced for classes of dynamical systems (Abate and Tiwari, 2006). The main motivation for this notion comes from biological case studies and models that were investigated with the following question: is the strict concept of Lyapunov stability always descriptive and computationally feasible in the biological realm, which is often characterized by complex dynamics with imprecisely known parameters? As a viable alternative, would it not make sense to look for bounded behavior in the closeness of an equilibrium? Intuitively, it is appealing to consider a notion of stability defined "within certain limits", or "bounds", rather than on ϵ-neighborhoods of equilibria. In other words, we may be interested in the existence of regions around the equilibrium within which a trajectory would indefinitely dwell once it reaches them. We focus here on the simplest possible shape for these regions–that of a box. We have found that models of many natural systems have box shaped regions as invariant sets. Box invariance is also appealing from the automated verification standpoint, since it is computationally tractable for a large class of systems and helps in proving strong safety properties of systems. The notion is related to other concepts in the literature, see Abate and Tiwari (2006) for details.

In this paper, we first formally define the notion of box invariance for continuous dynamical systems (Sec. 2). We then provide characterizations for linear and affine dynamical systems and show how to practically compute an actual box (Sec. 3). A detailed study of affine systems and full proofs of claims are presented elsewhere (Abate and Tiwari, 2006). The main focus of this article is on extending the notion to hybrid and switched systems (Sec. 4). These systems provide a powerful modeling tool, particularly for domains such as biology. As an illustrative example, we shall discuss a switched model for blood glucose concentration in human brain (Sec. 5) and analyze it using the concept of box invariance.

2. THE CONCEPT OF BOX INVARIANCE

We consider general, autonomous and uncontrolled dynamical systems of the form $\dot{\boldsymbol{x}} = f(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^n$. We assume basic continuity and Lipschitz properties for the existence of a unique solution of the vector field, given any possible initial condition. A rectangular box around a point $\boldsymbol{x}_0$ can be specified using two diagonally opposite points $\boldsymbol{x}_{lb}$ and $\boldsymbol{x}_{ub}$, where $\boldsymbol{x}_{lb} < \boldsymbol{x}_0 < \boldsymbol{x}_{ub}$ (interpreted component-wise). Such a box has $2n$ surfaces $S^{j,k} (1 \leq j \leq n, k \in \{l,u\})$, where

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$S^{j,k} = \{\boldsymbol{y} : (\boldsymbol{x}_{lb})_i \le \boldsymbol{y}_i \le (\boldsymbol{x}_{ub})_i$ for $i \ne j, \boldsymbol{y}_j = (\boldsymbol{x}_{lb})_j$ if $k = l, \boldsymbol{y}_j = (\boldsymbol{x}_{ub})_j$ if $k = u\}$.

***Definition** 1.* A dynamical system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ is said to be *box invariant* around an equilibrium point $\boldsymbol{x}_0$ if there exists a finite rectangular box around $\boldsymbol{x}_0$, specified by $\boldsymbol{x}_{lb}$ and $\boldsymbol{x}_{ub}$, such that for any point $\boldsymbol{y}$ on any surface $S^{j,k}(1 \le j \le n, k \in \{l, u\})$ of this rectangular box, it is the case that $f(\boldsymbol{y})_j \le 0$ if $k = u$ and $f(\boldsymbol{y})_j \ge 0$ if $k = l$. The system will be said to be *strictly box invariant* if the last inequalities hold strictly.

***Remark** 1.* The concept of box invariance for a dynamical system requires the existence of an *invariant set* with a special (polyhedral) shape. In the case of linear systems, we shall see that this invariant set is also an *ω-limit set* (or a *domain of attraction*).

***Definition** 2.* A system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ is said to be *symmetrical box invariant* around the equilibrium $\boldsymbol{x}_0$ if there exists a point $\boldsymbol{u} > \boldsymbol{x}_0$ (interpreted component-wise) such that the system is box invariant with respect to the box defined by $\boldsymbol{u}$ and $(2\boldsymbol{x}_0 - \boldsymbol{u})$.

3. CHARACTERIZATION OF BOX INVARIANCE.

*3.1 **Linear Systems.***

Given a linear system and a box around its equilibrium point, the problem of checking if the system is box invariant with respect to the given box can be solved by verifying the condition only at the 2^n vertices of the box (instead of all points on all the faces of the box).

***Proposition** 1.* A linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x}, \boldsymbol{x} \in \mathbb{R}^n$ is box invariant if there exist two points $\boldsymbol{u} \in (\mathbb{R}^+)^n$ and $\boldsymbol{l} \in (\mathbb{R}^-)^n$ such that for each point $\boldsymbol{c}$, with $c_i \in \{u_i, l_i\}, \forall i$, we have $A\boldsymbol{c} \ \sim \ \boldsymbol{0}$, where $\sim_i$ is $\le$ if $c_i = u_i$ and $\sim_i$ is $\ge$ if $c_i = l_i$.

Proposition 1 shows that box invariance of linear systems can be checked by testing satisfiability of 2^n linear inequality constraints (over $2n$ unknowns given by $\boldsymbol{l}$ and $\boldsymbol{u}$). In two steps, we will show that these 2^n constraints are subsumed by just n linear inequality constraints (over n unknowns). First we prove this fact for symmetric box invariance.

***Theorem** 1.* An n-dimensional linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ is symmetrical box invariant iff there exists a positive vector $\boldsymbol{c} \in {\mathbb{R}^+}^n$ such that $A^m\boldsymbol{c} \le \boldsymbol{0}$, where $a^m_{ii} = a_{ii} < 0$ and $a^m_{ij} = |a_{ij}|$ for $i \ne j$. This is equivalent to checking if the linear system defined by modified matrix A^m is symmetrical box invariant.

In the second step, we show the surprising result that the property of box invariance and that of symmetrical box invariance are equivalent for linear systems.

***Theorem** 2.* A linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$, where $A \in \mathbb{R}^{n\times n}$, is box invariant iff it is symmetrical box invariant.

Putting together Theorem 1 and Theorem 2, we conclude that to check if a linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ is box invariant, we only need to test if the set of n linear inequality constraints, succinctly written as $A^m\boldsymbol{c} \le 0$ (over the n unknowns $\boldsymbol{c}$) has a nonzero solution. This can be done in *polynomial* time. We can also find a box by generating solutions for the above linear constraint satisfaction problem. In general, it is possible to associate with a dynamical system, defined by system matrix $A \in \mathbb{R}^{n\times n}$, a *cone* in the positive $2^{n^{th}}$-ant described by the set

$$\mathcal{C} = \{\boldsymbol{x} \in {\mathbb{R}^+}^n : A^m\boldsymbol{x} \le \boldsymbol{0}\}.$$

Any choice of a single point or a pair of distinct points in $\mathcal{C}$ determine, respectively, a symmetric and a non-symmetric box for the system described by A. Box Invariance is a stronger notion than stability for linear systems.

***Theorem** 3.* If a linear dynamical system is box invariant around its equilibrium, then it is stable.

Surprisingly, the well-known concept of P-matrices in linear algebra provides a structural characterization for box invariant linear systems. A matrix A is said to be a *P-matrix* if all of its principal minors are positive.

***Theorem** 4.* Let A be a $n \times n$ matrix such that $a_{ii} \le 0$ and $a_{ij} \ge 0$ for all $i \ne j$. Then, the following statements are equivalent:

(1) The linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ is strictly symmetrical box invariant.
(2) $-A$ is a *P-matrix*.
(3) For every $i = 1, 2, \ldots, n$, the determinant of the top left $i \times i$ submatrix of $-A$ is positive.

***Remark** 2.* Theorem 4 shows that box invariance of general linear systems can also be tested by checking if the modified matrix $-A^m$ is a P-matrix. It is known that the problem of deciding if a given matrix is a P-matrix is co-NP-hard (Coxson, 1994; Coxson, 1999). Our case is special though, since we know that only the diagonal entries in $-A^m$ are positive. As a result, we can determine if $-A^m$ is a P-matrix using a simple *polynomial* time algorithm; for example, the Fourier-Motzkin elimination method can determine satisfiability and generate the cone $\mathcal{C}$.

Matrices with the shape of those in Theorem 4 (or, equivalently, of A^m in Theorem 1) have actually been studied under the appellative of *Metzler* matrices. There is a wealth of literature on Metzler matrices that can be used to derive results equivalent to those presented above for box invariance; see Abate and Tiwari (2006) for details.

***Remark** 3.* The concept of box invariance, which is closely related to that of classical stability in the linear case, can also be studied via Lyapunov arguments. In our particular instance, to prove box invariance we find a Lyapunov functional which is defined (at least) inside a certain boxed region of the state space. To go from smooth Lyapunov functions to one that is defined on a box, we can intersect proper ellipsoidal functions that have been adequately stretched to the limit (Abate and Tiwari, 2006) or use vector norms (Kiendl *et al.*, 1992). A linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ is box stable iff there exists a diagonal matrix $W \in (\mathbb{R}^+)^{n\times n}$ s.t. $V(\boldsymbol{x}) = ||W\boldsymbol{x}||_\infty$ is a Lyapunov function (cf. Kiendl *et al.* (1992)). The Lyapunov arguments will help in the case of hybrid and switched systems, as we shall see later.

3.2 ***Affine Systems.***

Consider the affine system, $\dot{\boldsymbol{x}} = A\boldsymbol{x}+\boldsymbol{b}$. We relate the box invariance of this system to the positivity of its equilibrium point, $\boldsymbol{x}_0 > \mathbf{0}$. The assumption of $\boldsymbol{x}_0$ being in the positive quadrant is justified both from a technical standpoint and from our applications.

Theorem *5.* If the affine system $\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{b}$ is s.t. A is Metzler and $\boldsymbol{b} > \mathbf{0}$, then its equilibrium point $\boldsymbol{x}_0 > \mathbf{0}$ iff the system is box invariant.

Remark *4.* The assumptions of the previous theorem can be relaxed to having a non-negative $\boldsymbol{b} \geq \mathbf{0}, \boldsymbol{b} \neq \mathbf{0}$.

In modeling biochemical pathways, the state variables $\boldsymbol{x}$ represent concentration of species such as proteins. When modeling the dynamics of such species, the Metzler form arises naturally since species decay proportionally to their concentration and are created proportionally to their precursor's concentrations. The vector $\boldsymbol{b} > 0$ represents the process of species creation (by transcription and translation, for example). For stable systems, such models typically have positive equilibrium point. By Theorem 5, all such models will be box invariant. Thus, Theorem 5 explains, in part, why many models proposed by biologists tend to be box invariant. For more results on box invariance of affine systems, the reader is referred to Abate and Tiwari (2006).

4. BOX INVARIANCE FOR HYBRID AND SWITCHED SYSTEMS.

In this Section we extend the notion of box invariance to the case of composition of different dynamical systems. We first define hybrid and switched systems and refer the reader to specialized literature for more details (Lygeros *et al.*, 2003; Ames *et al.*, 2005).

Definition *3.* A *hybrid system* is a tuple $\mathcal{H} = (Q, E, D, G, R, F)$, where

- $Q = \{1, ..., m\}$ is a finite set of *discrete states*.
- $E \subset Q \times Q$ is a set of *edges* that defines a source-target relation between the domains.
- $D = \{D_i\}_{i\in Q}$ is a set of *domains* where D_i is a compact subset of $\mathbb{R}^n$.
- $G = \{G_e\}_{e\in E}$ is a set of *guards*, where $G_e \subseteq D_{e(1)}$.
- $R = \{R_e\}_{e\in E}$ is a set of *reset maps*; here we assume identity resets.
- $F = \{f_i\}_{i\in Q}$ is a set of *vector fields* such that f_i is Lipschitz on $\mathbb{R}^n$.

A hybrid trajectory (in the state space) is described by adequately specifying a sequence of its initial conditions, switching times and edges, properly related via the guards and reset maps. The behavior of $\mathcal{H}$ allows for possible Zeno trajectories. As the reader may notice, the switching conditions (the "events") are due to spacial restrictions on the various domains. In contrast, *switched systems* specify jumping conditions in time, rather than in the state space.

Definition *4.* A *switched system* is a tuple $\mathcal{S} = (Q, E, D, G, R, F)$, where

- Q, E, R, F are characterized as in Def. 3.
- $D = \{D_i\}_{i\in Q}$ is a set of *domains* where $D_i = \mathbb{R}^n$.
- $G = \{0, \tau_1, \tau_2, \ldots\}$ is a set of *guards* in time, where $\tau_i \in \mathbb{R}^+$ are increasing in i. Each τ_i is mapped to a state by a function $g : G \to Q$ such that $(g(\tau_{i-1}), g(\tau_i)) \in E$ for all i.

Consider the hybrid domain $Q\times\mathbb{R}^n$. An *invariant set* of a hybrid or switched system is a subset of $Q \times \mathbb{R}^n$ such that every trajectory originating in this set continues to dwell inside it. The notion of box invariance for hybrid and switched cases is defined so as to emcompass the possibility of multiple equilibria and switchings between the different domains.

Definition *5.* A hybrid system $\mathcal{H}$ (a switched system $\mathcal{S}$) is said to be box invariant if there exists a boxed region $B \subset \mathbb{R}^n$ and a subset $Q' \subset Q$ of states such that $Q' \times B$ is an invariant set for $\mathcal{H}$ (or $\mathcal{S}$).

In this paper, we will restrict ourselves to linear vector fields, that is, all f_i's are linear functions of $\boldsymbol{x}$. Hence, in individual domains, boxes will be around the local equilibrium point. For the sake of analysis, we differentiate the following two cases: domains with different equilibria and domains with overlaying equilibria. Furthermore, in the hybrid case, the equilibria may or may not belong to one or more guard sets.

For a hybrid system $\mathcal{H}$, if some discrete state i has an equilibrium $\boldsymbol{x}_0$ that does not belong to any guard, that is $\boldsymbol{x}_0 \notin \bigcup_{e\in E} G_e$, then a sufficient condition for box invariance of $\mathcal{H}$ is obtained from box invariance of the dynamical system of state i.

In such a case, $\mathcal{H}$ is box invariant if there is a small enough box B for the dynamical system of state i that is completely contained in the domain, $B \subset D_i$, and that does not intersect the guard, $B \cap \bigcup_{e \in E} G_e = \emptyset$. This occurs, for example, in the hybrid model of the Delta-Notch lateral inhibition mechanism of Ghosh and Tomlin (2001). The reader should notice that, as in the single-domain nonlinear case, the existence of a box does not imply the existence of boxes of different sizes: expanding a box may cause it to intersect a guard. Next consider the case where $\boldsymbol{x}_0$ is a shared equilibrium that belongs to at least one spacial guard. In this instance, a sufficient condition is the existence of a *single* box for *multiple* domains.

For switching systems, both in the case of shared equilibrium and different equilibria, jumps at possibly any time instant forces us to find a *common* box, which would be invariant in all the domains.

The results in this section resemble those obtained for stability of hybrid and switched systems. In particular, it is known that there are examples of unstable hybrid and switched systems that are the composition of *stable* dynamical systems (Branicky, 1994). We are similarly interested in understanding how the notion of box invariance, which is shown in the linear case to be intrinsically related to that of stability, is translated in the hybrid or switched setting. Furthermore, sufficient conditions for the invariance of the interconnection, according to Def. 5, will be derived. To begin with, let us stress a technique to obtain sufficient conditions for a hybrid dynamical system.

***Proposition** 2.* Let us associate to a hybrid system $\mathcal{H}$ a corresponding switched system $\mathcal{S}$, made up of the same tuple, except for the following two elements: the domains $D_i = \mathbb{R}^n$ and a symbolic non decreasing sequence $G = \{0, \tau_1, \tau_2, \ldots\}$. Given an initial condition, if a universal property $\mathcal{P}$ holds in all trajectories of $\mathcal{S}$ (given by all the possible different sequences with the form of G), then $\mathcal{P}$ holds in $\mathcal{H}$.

The following result deals with the case of switched linear systems. Such systems share the origin as a common equilibrium.

***Theorem** 6.* A switched linear system $\mathcal{S}$, characterized by a set of vector fields of the form:

- $F = \{f_i\}_{i \in Q} = A_i \boldsymbol{x},\ i \in Q;$

is box invariant around the origin if there exists a single box around which each of the dynamical systems is box invariant. Thus, a sufficient condition for box invariance is that $\bigcap_{i \in Q} \mathcal{C}_i \neq \emptyset$, where the cones $\mathcal{C}_i$'s are defined as in Section 3.1.

Checking whether two cones, both pivoted on the origin, intersect is equivalent to checking for the intersection of two convex sets. We need to find a $\boldsymbol{c}$ such that $A_i \boldsymbol{c} \leq 0$ for all $i \in \{1, \ldots, m\}$. This is a linear program that can be solved in time polynomial in nm.

***Remark** 5.* There is a lot of work (for instance, see Branicky (1994)) in describing conditions for the Lyapunov stability of switched systems. These conditions either assume the existence of a common Lyapunov function, or require the presence of multiple Lyapunov functions (one for each domain) with certain switching restrictions. Our result can be interpreted as follows: a common box can be thought of as being the equivalent of a "common Lyapunov function" (the observations developed in Remark 3 should make this point clear). The explicit computation of a global Lyapunov function reduces to solving a set of linear matrix inequalities. Many efforts have been made to ease this task (Johansson and Rantzer, 1998). Otherwise, we may intuitively want to come up with conditions on the possible switchings by allowing the trajectory to jump between domains only when it belongs to the intersection of the boxes of these domains: this condition unfortunately seems harder to impose.

Handling box invariance for hybrid systems with general spacial guards is not easy, as we shall later discuss (see for instance Ex. 1). Nevertheless, exploiting the fact that box invariance is a stronger property than stability, we report the following compositional result for proving stability of hybrid systems (illustrated in Fig. 1).

***Theorem** 7.* If a linear hybrid system is composed of box invariant linear systems with a single spacial guard described by a hyperplane crossing the shared equilibrium, then the hybrid system is stable.

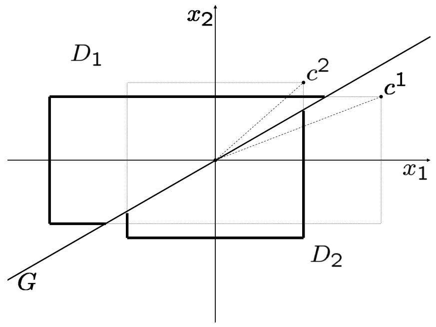

Fig. 1. A planar hybrid system with one single guard (line), as in Th. 7.

The previous theorem holds under rather restrictive conditions. Its generalization to the case of hybrid systems with multiple guards per domain does not hold.

***Example** 1.* Consider the following two-dimensional HS characterized by two modes with domains coinciding with the whole space, $D_1 = D_2 = (\mathbb{R}^+)^2$, and endowed with the following vector fields:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = A_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} -1 & 5 \\ -0.1 & -1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix};$$
$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = A_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} -1 & -0.2 \\ 4 & -1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Assume that there are two edges with the following guards in $\mathbb{R}^2$:

$$G_{1\to 2}(x_1, x_2) = \{\boldsymbol{x} \in \mathbb{R}^2 : x_1 - 5x_2 = 0\};$$
$$G_{2\leftarrow 1}(x_1, x_2) = \{\boldsymbol{x} \in \mathbb{R}^2 : 4x_1 - x_2 = 0\}.$$

Assume again trivial reset maps, and initial condition $(x_1(0), x_2(0)) = (0.1, 0.1) \in D_1$.
In isolation, both linear systems are box invariant and indeed have spiraling convergent trajectories towards the origin. The HS though is evidently unstable (see Fig. 2). Notice that

$$\mathcal{C}_1 = \{(x_1, x_2) : x_1 - 5x_2 \geq 0 \wedge x_2 \geq 0\};$$
$$\mathcal{C}_2 = \{(x_1, x_2) : 4x_1 - x_2 \leq 0 \wedge x_1 \geq 0\};$$

and that $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. □

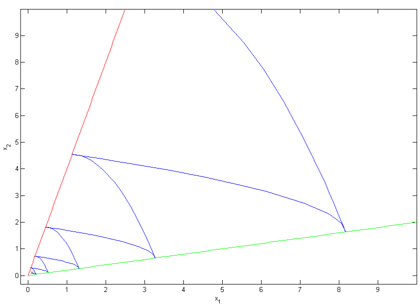

Fig. 2. Simulation for Example 1.

We finally consider the case of affine hybrid (and switched) systems. For a system of the form $\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{b}$ and equilibrium $\boldsymbol{x}^{eq} = -A^{-1}\boldsymbol{b}$, we transform the variables into the new set $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{x}^{eq}$ and consider the new system $\dot{\boldsymbol{y}} = A\boldsymbol{y}$. Using the modified matrix A^m, we define $\mathcal{C} = \{\boldsymbol{y} \in \mathbb{R}^{+^n} : A^m\boldsymbol{y} \leq \boldsymbol{0}\}$. Let us introduce the "negative cone" $\mathcal{C}^- = \{\boldsymbol{y} \in \mathbb{R}^{-^n} : (-A^m)\boldsymbol{y} \leq \boldsymbol{0}\}$. Note that the actual cones, $\mathcal{C} + \boldsymbol{x}^{eq}$ and $\mathcal{C}^- + \boldsymbol{x}^{eq}$, are obtained by translating these cones to the equilibrium point.

Unlike the linear case, each subsystem could have a different equilibrium point in an affine hybrid system. However, we can still derive sufficient conditions for the existence of a "common box" as in the case of linear hybrid and switched systems (see Fig. 3).

***Theorem** 8.* Consider an affine hybrid system $\mathcal{H}$, where each domain has an equilibrium point $\boldsymbol{x}_i^{eq}, i \in Q$, and all variables are bound to be positive. $\mathcal{H}$ is box invariant if $\bigcap_{i\in Q}(\mathcal{C}_i + \boldsymbol{x}_i^{eq}) \neq \emptyset$.

More generally, if we allow the state space to include negative values, we need to make the condition stronger. Now, the existence of a global box

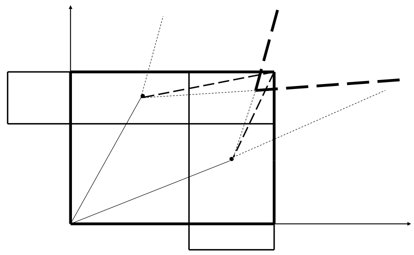

Fig. 3. Common box (shown in dark) for a two mode positive affine switched system.

will depend on having a nonempty intersection of both the positive and negative cones.

***Theorem** 9.* Consider an affine hybrid system $\mathcal{H}$, where each domain has an equilibrium point $\boldsymbol{x}_i^{eq}, i \in Q$. $\mathcal{H}$ will be box invariant if the following holds:

$$\bigcap_{i\in Q}(\mathcal{C}_i + \boldsymbol{x}_i^{eq}) \neq \emptyset \bigwedge \bigcap_{i\in Q}(\mathcal{C}_i^- + \boldsymbol{x}_i^{eq}) \neq \emptyset.$$

In conclusion, the concept of box invariance, even though stronger than that of asymptotic stability, is still not fully compositional in the hybrid case. Nevertheless, the sufficient conditions we have proposed for composability are easy to check and quite general. The comparison with the literature on stability of hybrid and switched systems confirms this fact. Furthermore, our sufficient conditions are enough to establish box invariance in many applications, one of which is described next.

5. A MODEL FOR BLOOD GLUCOSE CONCENTRATION

The following model is taken from (Sorensen, 1985). It is a model of a physiologic compartment, specifically the human brain, and describes the dynamics of the blood glucose concentration. In general, this compartment is part of a network of different parts, which model the concentration in other organs of the body, and which follow some conservation laws that account for the exchange of matter between different compartments. The mass balance equations are the following:

$$V_B\dot{C}_{Bo} = Q_B(C_{Bi} - C_{Bo}) + PA(C_I - C_{Bo}) - r_{RBC}$$
$$V_I\dot{C}_I = PA(C_{Bo} - C_I) - r_T,$$

where V_B describes the capillary volume, V_I the interstitial fluid volume, Q_B the volumetric blood flow rate, PA the permeability-area product, C_{Bi} the arterial blood solute concentration, C_{Bo} the capillary blood solute concentration, C_I the interstitial fluid solute concentration, r_{RBC} the rate of red blood cell uptake of solute, and r_T models the tissue cellular removal of solute through cell membrane. The quantity PA can be expressed as the ratio V_I/T, where T is the transcapillary diffusion time.

			T	10 or 3 [min]	
V_B	0.04 [l]		V_I	0.45 [l]	
Q_B	0.7 [l/min]		C_{Bi}	0.15 [kg/l]	
r_T	2×10^{-6} [kg/min]		r_{RBC}	10^{-5} [kg/min]	

We allow T to assume two different values, which correspond to different physiological situations; furthermore, we assume that the switch between these two conditions can happen at any time. This calls for the introduction of a bimodal switched model, composed by the following two dynamics:

$$\dot{\boldsymbol{x}} = A_1\boldsymbol{x} + b, \text{ if T} = 10[\text{min.}];$$
$$\dot{\boldsymbol{x}} = A_2\boldsymbol{x} + b, \text{ if T} = 3[\text{min.}].$$

It can be easily checked that both models, considered in isolation, are box invariant. They therefore are stable around two different equilibria. In Fig. 4 we plot trajectories for these two systems, and draw some boxes (that were obtained from the computation of the eigenvalue with the rightmost real part and its corresponding eigenvector). Additionally, pivoted on the equilibria, the cones are shown. In Fig. 5 the two realizations are shown superimposed: the intersection of the two cones is non empty (the cones differ by only one of their boundaries, as expected because of the relative modification of only one row of the state matrix). The smaller one was thus chosen to define the "global" box. Two different simulations, with random switching, starting from opposite initial conditions, are shown in Fig. 6. As can be observed, the box is indeed an invariant for the switched system. This box yields a bound on the values of $\boldsymbol{x}$, which is the brain blood glucose concentration. These bounds are useful to verify safety of insulin treatments for models of Type I diabetes patients.

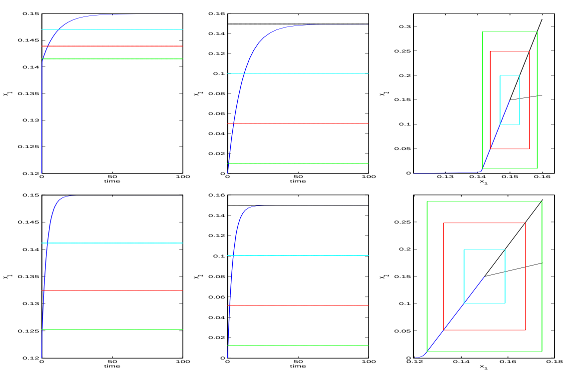

Fig. 4. Simulation of a trajectory for the first(top part) and second(bottom part) system, and computation of some symmetrical boxes.

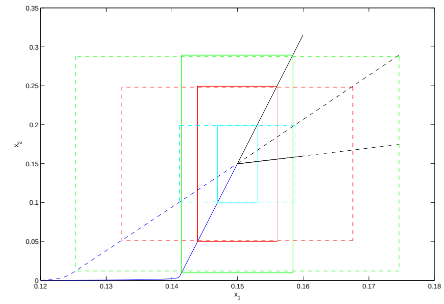

Fig. 5. Comparing the two trajectories, their respective boxes and conical regions. The first in solid, the second in dashed lines.

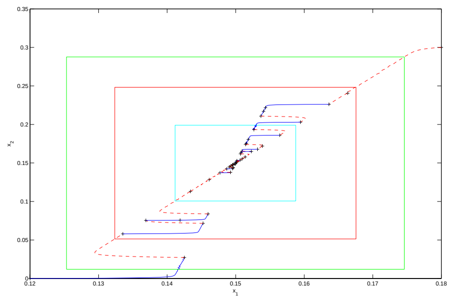

Fig. 6. Randomly Switching System composed by the two dynamical systems, with global boxes.

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