Deciding Fundamental Properties of Right-(Ground or Variable) Rewrite Systems by Rewrite Closure *

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Abstract. Right-(ground or variable) rewrite systems (RGV systems for short) are term rewrite systems where all right hand sides of rules are restricted to be either ground or a variable. We define a minimal rewrite extension $\overline{R}$ of the rewrite relation induced by a RGV system R. This extension admits a rewrite closure presentation, which can be effectively constructed from R. The rewrite closure is used to obtain decidability of the reachability, joinability, termination, and confluence properties of the RGV system R. We also show that the word problem and the unification problem are decidable for confluent RGV systems. We analyze the time complexity of the obtained procedures; for shallow RGV systems, termination and confluence are exponential, which is the best possible result since all these problems are EXPTIME-hard for shallow RGV systems.

1 Introduction

It is being increasingly realized that theorem provers are most effective and useful in their incarnation as specialized decision procedures for restricted logics. This is true, in particular, for equational theories, where rewriting techniques have provided effective decision procedures. In the simplest case of equality between constants, rewrite rules, representing a Union-Find data structure, decide the word problem. At the next level, flat rewrite rules, in the form of abstract congruence closure, handle ground equational theories. This can be generalized to shallow-linear equational theories.

In going from special to more general equational theories, the complexity of deciding various fundamental problems, like the word problem, termination, confluence, reachability, and joinability, increases until all these problems become undecidable. It is, therefore, fruitful to know how far the approach towards "specialized theorem provers" can take us. In this context, we consider RGV term rewrite systems, where every rule $l \to r$ is such that r is either a ground term or a variable (and there are no restrictions on l). For example, the rules $0 + x \to x,\ x + 0 \to x,\ x + (-x) \to 0,\ (-x) + x \to 0$, and $-(-x) \to x$ are all of this kind. It is known that the word

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problem for this class of term rewriting systems is undecidable [12]. In this paper, we show that the reachability, joinability, termination, and confluence properties are decidable. We also show that the word problem is decidable for confluent RGV systems. We note here that most of these properties are undecidable even for very restricted classes of term rewrite systems [9].

Takai, Kaji, and Seki [14] showed that right-linear finite-path-overlapping systems effectively preserve recognizability. Since RGV systems are right-linear and finite-path-overlapping, it follows that the reachability and joinability problems for RGV systems are decidable. We obtain the same results using a different approach. Takai, Kaji, and Seki [14] construct a tree-automaton that represents all terms reachable from a given term, whereas we construct a rewrite closure (which works for all initial choices of terms). The generality of rewrite closure allows us to obtain new decidability results for confluence and termination. We believe that, apart from the main decidability results, our approach to obtaining them is also significant. We build on the simple cases in a modular way so that an implementation can seamlessly use efficient implementation of, say, the union-find data structure for handling equality between constants and congruence closure for ground theories (which itself uses the union-find as a subroutine). This is achieved by generalizing the existing definitions minimally, for example, *flat rules* of the form $fc_1 \dots c_m \to c$, are generalized to richer F-rules. Second, we make specialized use of more general concepts so that it is easy to see where further generalizations to larger classes of rewrite system fail. For instance, we use the canonical concept of rewrite closure, in parallel to congruence closure or convergent presentations, for nonsymmetric properties such as reachability. We show that RGV systems admit rewrite closure presentation, but only after we have introduced a new concept of minimal rewrite extensions $\overline{R}$.

The results described in this paper build upon our previous works [7, 8]. Some of the techical proofs, which are easy extensions of previously published proofs, have been left out. The extension to RGV systems was motivated by the observation that several important axioms can be stated using collapsing rules (i.e. right variable rules). In our previous work, rewrite closure construction has been a crucial first step for deciding some properties of term rewrite systems. Intuitively, a rewrite closure for a rewrite system R is the union of two rewrite systems $F \cup B$ such that the relations induced by R and by $F \cup B$ coincide, and rewriting with $F \cup B$ can be always re-ordered to rewrite first with only F, and then with only B; moreover, F is (size-)decreasing whereas B is (size-)nondecreasing.

The following example introduces the key ideas in the construction of a rewrite closure for RGV rewrite systems. Consider the rewrite system $R = \{c_1 \to fc_1\ ,\ c_2 \to fc_2\ ,\ c_1 \to fc'_1\ ,\ c_2 \to fc'_2\ ,\ c'_1 \to fc_3\ ,\ c'_2 \to fc_3\ ,\ gxx \to x\}$ (we use the notation fc_1 to represent $f(c_1)$ and gxx to represent $g(x,x)$). Note that R contains a size-decreasing rule $gxx \to x$ and many size-increasing rules. The term gc_1c_2 rewrites into fc_3 by R: $gc_1c_2 \to_{c_1 \to fc'_1} gf(c'_1)c_2 \to_{c_2 \to fc'_2} gf(c'_1)f(c'_2) \to_{c'_1 \to fc_3} gf(fc_3)c'_2 \to_{c'_2 \to fc_3} gf(fc_3)f(fc_3) \to_{gxx \to x} f(fc_3)$. But the first four rewrite steps are increasing and the last one is decreasing, and hence, this derivation is not of the required form. In order to construct a rewrite closure for R, we replace the rule $gxx \to x$ by the constrained rule $gx_1x_2 \to x_3$ **if** $\{x_3; x_1, x_2\}$. A substitution σ is a solution to the constraint $\{x_3; x_1, x_2\}$ if $x_3\sigma$ is reachable from $x_1\sigma$ and $x_2\sigma$. Hence, this new rule represents all the rules $gs_1s_2 \to s_3$ such that s_3 is reachable from s_1 and s_2 by the increasing rules; and in particular, $gc_1c_2 \to f(fc_3)$. But these new rules are not necessarily decreasing, and consequently we cannot claim that we have a rewrite closure. The other ingredient we need is the introduction of new constants like $c_{\{1,2\}}$ that represent all the terms reachable from c_1 and c_2. Hence, we need to add new rules like $c_{\{1,2\}} \to fc_{\{1',2'\}}$ and $c_{\{1',2'\}} \to fc_3$, and now, we have the derivation $gc_1c_2 \to_{gx_1x_2 \to x_3 \textbf{ if } \{x_3;x_1,x_2\}} c_{\{1,2\}} \to_{c_{\{1,2\}} \to fc_{\{1',2'\}}} fc_{\{1',2'\}} \to_{c_{\{1',2'\}} \to fc_3} f(fc_3)$ of the required form.

Outline of the paper. Section 2 introduces some basic notions and notations. In Section 3 we argue that we can make some simplifying assumptions on the initial RGV system R without loss of generality. These will simplify the arguments in the rest of the paper. Section 4 introduces the concept of minimum joining extension $\overline{R}$ of a rewrite system R, and shows that this extension preserves some properties of the original R. Section 5 defines the concept of constrained rewriting with joinability constraints. In Section 6 we construct a rewrite closure for $\overline{R}$. As a consequence, we obtain the decidability of reachability and joinability for RGV systems. In Section 7 we prove the decidability of termination for RGV systems. In Section 8 we give a characterization of the confluence of RGV systems, and use it to prove the decidability of confluence for RGV systems, and the decidability of the word problem for confluent RGV systems. In Section 9 we briefly analyse the complexity of all the presented algorithms.

2 Preliminaries

We use standard notation from the term rewriting literature. A signature Σ is a (finite) set of function symbols, which is partitioned as $\cup_i \Sigma_i$ such that $f \in \Sigma_n$ if arity of f is n. Symbols in Σ_0, called *constants*, are denoted by a, b, c, d, with possible subscripts. The elements of a set $\mathcal{X}$ of variable symbols are denoted by x, y with possible subscripts. The set $\mathcal{T}(\Sigma, \mathcal{X})$ of *terms* over Σ and $\mathcal{X}$, *position* p in a term, *subterm* $t|_p$ of term t at position p, and the term $t[s]_p$ obtained by replacing $t|_p$ by s are defined in the standard way. For example, if t is $f(a, g(b, h(c)), d)$, then $t|_{2.2.1} = c$, and $t[d]_{2.2} = f(a, g(b, d), d)$. By $t[s_1, s_2, \ldots, s_n]_{p_1,p_2,\ldots,p_n}$ we denote $t[s_1]_{p_1}[s_2]_{p_2} \ldots [s_n]_{p_n}$. By $Pos(t)$ we denote the set of all positions p such that $t|_p$ is defined. By $Vars(t)$ we denote the set of all variables occurring in t. The *height* of a term s is 0 if s is a variable or a constant, and $1 + max_i height(s_i)$ if $s = f(s_1, \ldots, s_m)$. Usually we will denote a term $f(t_1, \ldots, t_n)$ by the simplified form $ft_1 \ldots t_n$, and $t[s]_p$ by $t[s]$ when p is clear by the context or not important.

A substitution σ is sometimes presented explicitly as $\{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$. We assume standard definition for a *rewrite rule* $l \to r$, a *rewrite system* R, the *one step rewrite relation at position p induced by* $R \to_{R,p}$, and the *one step rewrite relation induced by* R (at any position) $\to_R$. If R is any rewrite system, then R^- denotes the rewrite system $\{r \to l \mid l \to r \in R\}$. If $p = \lambda$, then the rewrite step $\to_{R,p}$ is said to be applied *at the topmost position* (at the root) and is denoted by $s \to_R^r t$; it is denoted by $s \to_R^{nr} t$ otherwise. With $s \to_R^{||} t$ we denote a derivation $s \to_R^* t$ where all the rules are applied at disjoint positions, i.e. this derivation is not of the form $s \to_R^* \circ \to_{R,p} \circ \to_R^* \circ \to_{R,q} \circ \to_R^* t$ where p is a prefix of q or q is a prefix of p.

The notations $\leftrightarrow$, $\to^+$, and $\to^*$, are used respectively for the symmetric, transitive, and reflexive-transitive closure of a binary relation $\to$. A rewrite system R is *confluent* if the relation $\leftarrow_R^* \circ \to_R^*$ is contained in $\to^* \circ \leftarrow^*$, which is equivalent to the relation $\leftrightarrow_R^*$ being contained in $\to^* \circ \leftarrow^*$ (called the Church-Rosser property). A term t is *reachable* from s by R if $s \to_R^* t$. A term s is *R-irreducible* (or, in R-normal form) if there is no term t such that $s \to_R t$. We denote by $s \to_R^! t$, or $t = NF_R(s)$, the fact that an R-irreducible term t is reachable from s by R. Two terms s and t are said to be *equivalent* by R (or, R-equivalent) if $s \leftrightarrow_R^* t$. The terms s and t are *R-joinable*, denoted by $s \downarrow_R t$, if $s \to_R^* \circ \leftarrow_R^* t$. A *(rewrite) derivation or proof* (from s) is a sequence of rewrite steps (starting from s), that is, a sequence $s \to_R s_1 \to_R s_2 \to_R \ldots$.

A TRS R is *terminating* if there is no infinite derivation $s \to_R s_1 \to_R s_2 \to_R \cdots$. Termination is usually ensured by showing that the rewrite system R is contained in a *reduction ordering* [5]. A confluent and terminating term rewrite system is said to be *convergent*.

A term t is called *ground* if t contains no variables. A term rewrite system R is *right-(ground or variable)* (RGV) if for every $l \to r \in R$, the term r is either ground, or a variable.

3 Simplifying assumptions on the initial RGV system

Let R be a RGV term rewrite system over the signature Σ. We henceforth assume that a left hand side of a rule in R is not a variable. If this were the case, R would be trivially not terminating, and either it would be trivially confluent or this rule would be useless (of the form $x \to x$). For reachability and joinability, the procedures introduced here can be easily adapted to handle rules of this kind.

We assume, without loss of generality, that (a) the rewrite system R is over the signature $\Sigma = \Sigma_0 \cup \Sigma_m$, where $\Sigma_0 = \{c_1 \ldots c_n\}$ and $\Sigma_m = \{f\}$, and (b) R is partitioned into $F \cup B$ in such a way that F contains rules of the form $fs_1 \ldots s_m \to \alpha$ (where α is a constant or a variable, or, in short, a height 0 term); whereas B contains rules of the form $c \to fc_1 \ldots c_m$ and rules of the form $c \to d$. The first assumption is made to simplify the presentation of proofs and is not crucial for correctness.

These transformations are standard and have been formally presented before [7, 8]. We will instead just give an example.

Example 1. Let $\Sigma_0 = \{a\}, \Sigma_1 = \{g\}$, and $\Sigma_2 = \{h\}$. Terms over the signature $\cup_i \Sigma_i$ can be mapped onto terms over a new signature $\Sigma_3' \cup \Sigma_0'$ where $\Sigma_3' = \{f\}$ and $\Sigma_0' = \{a, g, h\}$. The term a is mapped to $f(a, a, a)$, the term $g(x)$ to $f(x, g, g)$, and the term $h(a, g(x))$ to $f(f(a, a, a), f(x, g, g), h)$.

The second transformation is achieved by flattening right-hand side ground terms. For example, the rewrite rule $s \to f(f(a, a, a), f(a, g, g), h)$ is replaced by $s \to c_1$, while introducing the new B-rules $c_1 \to fc_2c_3h$, $c_2 \to faaa$, and $c_3 \to fagg$.

4 New Constants and the Minimum Joining Extension

Let R be a RGV system. Suppose $fxx \to x$ is a rewrite rule in R. We shall see in later sections that it is useful to interpret the rule $fxx \to x$ as representing all the instances $fs_1s_2 \to s_3$ where s_3 is a term simultaneously reachable from s_1 and s_2 by R. In general, define

$$Reach_R(\{s_1, \dots, s_k\}) = \{s : \forall i.s_i \to_R^* s\}.$$

We say that the set $\{s_1, \dots, s_k\}$ is R-joinable if $Reach_R(\{s_1, \dots, s_k\}) \neq \emptyset$.

The set $Reach_R(\{s_1, \dots, s_k\})$ is especially important when every s_i is a constant. Let $\{c_1 \dots c_n\}$ be the set of constants appearing in R. We introduce new constants of the form $c_{\{i_1 \dots i_k\}}$, where $\{i_1 \dots i_k\}$ is a subset of $\{1 \dots n\}$. Our intention is to make a term s reachable from $c_{\{i_1 \dots i_k\}}$ iff $s \in Reach_R(\{c_{i_1}, \dots, c_{i_k}\})$. For uniformity, we first rename every constant c_i to $c_{\{i\}}$; i.e. we assume that the set of constants appearing in R is $\{c_{\{1\}} \dots c_{\{n\}}\}$. The new constants are counterparts of the *packed states* in [14]. Let Σ' be $\Sigma \cup \{c_S \mid \emptyset \neq S \subseteq \{1 \dots n\}$.

Definition 1. *The* minimum joining extension*, $\overline{R}$, of a rewrite system R is defined as the minimum rewrite relation*[1] *on $\mathcal{T}(\Sigma', \mathcal{X})$ such that (a) $\to_R^* \subseteq \overline{R}$, and (b) for all subsets $S = \{i_1, \dots, i_k\} \subseteq \{1 \dots n\}$,*

$$Reach_{\overline{R}}(\{c_S\}) = \begin{cases} \{c_S\} & \textbf{if } Reach_{\overline{R}}(\{c_{\{i_1\}}, \dots, c_{\{i_k\}}\}) = \emptyset \\ Reach_{\overline{R}}(\{c_{\{i_1\}}, \dots, c_{\{i_k\}}\}) & \textit{otherwise} \end{cases}$$

Note that there exist rewrite relations that satisfy conditions (a) and (b); for example, the rewrite relation induced by $R \cup \{c_S \leftrightarrow c_{\{i\}} : \emptyset \neq S \subseteq \{1, \dots, n\}, i \in S\}$ is one such relation. If $\{R_i\}_{i \in I}$ are rewrite relations that satisfy conditions (a) and (b), then $\bigcap_{i \in I} R_i$ also satisfies the two conditions. Therefore, the *minimum* is well defined.

Example 2. If $R = \{c_{\{1\}} \to fc_{\{4\}},\ c_{\{2\}} \to fc_{\{4\}},\ c_{\{3\}} \to fc_{\{4\}}\}$, then $\overline{R}$ is the rewrite relation induced by $\{c_S \to c_{S'} : \emptyset \neq S \subseteq S' \subseteq \{1, 2, 3\}\} \cup \{c_{\{1,2,3\}} \to fc_{\{4\}}\}$.

We will use $Const(\{i_1 \dots i_k\})$ to denote the set $\{c_{\{i_1\}}, \dots, c_{\{i_k\}}\}$ and infix notation $s\overline{R}t$ to denote $(s, t) \in \overline{R}$.

Lemma 1. *The following properties are true about the the minimum joining extension $\overline{R}$ of R:*

1. *If $\emptyset \neq S \subseteq S'$ and $Reach_{\overline{R}}(Const(S')) \neq \emptyset$ then $c_S\ \overline{R}\ c_{S'}$.*
2. *If $Reach_{\overline{R}}(Const(S_1)) \neq \emptyset$ and $Reach_{\overline{R}}(Const(S_2)) \neq \emptyset$ and $\{c_{S_1}, c_{S_2}\}$ is $\overline{R}$-joinable, then $Reach_{\overline{R}}(Const(S_1 \cup S_2)) \neq \emptyset$.*
3. *If $c_{S_1}\ \overline{R}\ c_{S_1'}$, $c_{S_2}\ \overline{R}\ c_{S_2'}$, and the sets $Reach_{\overline{R}}(Const(S_1))$, $Reach_{\overline{R}}(Const(S_2))$, $Reach_{\overline{R}}(Const(S_1'))$ and $Reach_{\overline{R}}(Const(S_2'))$ are not empty, and and $\{c_{S_1'}, c_{S_2'}\}$ is $\overline{R}$-joinable, then $c_{S_1 \cup S_2}\ \overline{R}\ c_{S_1' \cup S_2'}$.*

[1] A rewrite relation is a reflexive, transitive, and monotonic relation that is also closed under substitution.

4. *If* $c_{S_1}\overline{R}fc_{S_{11}}\ldots c_{S_{1m}}$*, and* $c_{S_2}\overline{R}fc_{S_{21}}\ldots c_{S_{2m}}$*, and the sets* $\mathit{Reach}_{\overline{R}}(Const(S_{11}\cup S_{21}))$*,...,* $\mathit{Reach}_{\overline{R}}(Const(S_{1m}\cup S_{2m}))$ *are not empty, then* $c_{S_1\cup S_2}\overline{R}fc_{S_{11}\cup S_{21}}\ldots c_{S_{1m}\cup S_{2m}}$.

The following technical lemma provides an alternate characterization of $\overline{R}$.

Lemma 2. *Let* R_J *be* $\{c_S \to t[c_{S_1},\ldots,c_{S_l}]_P \mid \forall i \in S.\forall j \in \{1,\ldots,l\}.(c_{\{i\}} \to_R^* t[c_{\{k_{i1}\}},\ldots,c_{\{k_{il}\}}]_P \ \wedge\ k_{ij} \in S_j) \ \wedge\ \mathit{Reach}_R(Const(S_j)) \neq \emptyset)\}$*. Then* $\overline{R} = \to_{R_J}^* \circ \to_R^* \circ \to_{R_J}^*$.

The intuition that the minimum joining extension $\overline{R}$ only conservatively extends the rewrite relation R is captured by the following two corollaries.

Corollary 1. *Let* $\overline{R}$ *be the minimum joining extension of* R*. Then,*

- $\overline{R}$ *and* $\to_R^*$ *coincide on pairs of terms over the original signature.*
- *The set* $\mathit{Reach}_R(Const(S))$ *is nonempty iff* $\mathit{Reach}_{\overline{R}}(Const(S))$ *is nonempty iff* $\mathit{Reach}_{\overline{R}}(\{c_S\}) \neq \{c_S\}$.

Corollary 2. *The following is true for the minimum joining extension* $\overline{R}$ *of* R*:*

- *If* s *is a term over the original signature and* $s\overline{R}t$*, then for all constant* c_S *appearing in* t*,* $Const(S)$ *is* R*-joinable.*
- *If* s *is a term containing only constants* c_S *such that* $Const(S)$ *is* R*-joinable, then there exists a term* t *over the original signature such that* $s\overline{R}t$*.*
- *If* s *is a term over the original signature and* $s\overline{R}t$*, then there exists* t' *over the original signature such that* $t\overline{R}t'$*.*
- *Joinability of* $\overline{R}$ *coincides with joinability of* $\to_R^*$ *for terms over the original signature.*

5 Constrained Rewriting with Joinability Constraints

In the following sections we will work with a particular form of constrained rewriting where the constraints are interpreted modulo joinability by a given rewrite system.

Definition 2. *A* constrained rule *is of the form* $l \to r$ **if** $\mathcal{C}$*, where (i)* r *is ground or a variable, (ii) no variable appears twice in* $l \to r$*, and (iii)* $\mathcal{C}$ *is a set of subsets of* $\mathit{Vars}(l \to r) \cup \Sigma_0'$ *such that every variable appears in at most one subset; moreover, there is a distinguished element in one of these subsets if and only if this element is a variable and coincides with* r.

We will denote a subset with no distinguished element by $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ and a subset with a distinguished element α by $\{\alpha; \alpha_1, \alpha_2, \ldots, \alpha_k\}$.

The interpretation of constraints is dependent upon (the joinability relation induced by some) rewrite system R.

Definition 3. *Let* R *be a rewrite system. A substitution* σ *is an* R-solution *of the constraint* $\mathcal{C}$ *if for each* $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \in \mathcal{C}$*, the terms* $\alpha_1\sigma, \ldots, \alpha_k\sigma$ *are* R*-joinable, and for each* $\{\alpha; \alpha_1, \alpha_2, \ldots, \alpha_k\} \in \mathcal{C}$*, the term* $\alpha\sigma$ *is in* $\mathit{Reach}_R(\{\alpha_1\sigma, \ldots, \alpha_k\sigma\})$.

Thus, for a given R, a constrained rewrite rule $l \to r$ **if** $\mathcal{C}$ denotes all instances $l\sigma \to r\sigma$, where σ is an R-solution of $\mathcal{C}$.

Definition 4. *Given a set C of constrained rules, we write $s = u[l\sigma] \to_{C/R} u[r\sigma] = t$ if $(l \to r$ **if** $\mathcal{C})$ is a constrained rule in C and σ is an R-solution of $\mathcal{C}$.*

Example 3. If $C = \{(x_1 + (-x_2)) + x_3 \to x_4$ **if** $\{\{x_4; x_1, x_2, x_3\}\}\}$ and R is as in Example 2, then $(c_{\{1\}} + -c_{\{2\}}) + c_{\{3\}} \to_{C/R} c_{\{1,2,3\}}$, and $(c_{\{1\}} + -c_{\{2\}}) + c_{\{3\}} \to_{C/R} fc_{\{4\}}$.

6 Rewrite Closure

A pair of term rewrite systems (F, B) is a rewrite closure for R (with respect to a reduction ordering $\succ$) if (i) $F \cup B^-$ is contained in $\succ \cup (\Sigma_0 \times \Sigma_0)$ and (ii) the relation $\to_R^*$ is equivalent to the relation $\to_F^* \circ \to_B^*$. We will now construct a rewrite closure for the minimal extension $\overline{R}$ of a RGV system R.

First, we transform the rules in F to *linear* constrained rules by "moving" the non-linearity information into constraints. For example, the rule $(x + -x) + x \to x$ is transformed into the constrained rule

$$(x_1 + -x_2) + x_3 \to x_4 \qquad \textbf{if} \qquad \{\{x_4; x_1, x_2, x_3\}\}.$$

Intuitively, this constrained rewrite rule represents all instances $((x_1 + -x_2) + x_3)\sigma \to x_4\sigma$, where σ is such that $x_4\sigma \in Reach_R(\{x_1\sigma, x_2\sigma, x_3\sigma\})$.

Definition 5. *A* constrained form *of an F-rule $l \to r$ is a constrained rule $l' \to r'$* **if** $\mathcal{C}$ *where (i) $\mathcal{C}$ is a partition of $Vars(l') \cup Vars(r')$, and (ii) there is a mapping $\sigma : Vars(l') \cup Vars(r') \to Vars(l) \cup Vars(r)$ such that $l'\sigma = l$, $r'\sigma = r$ and $x\sigma = y\sigma$ iff x and y are in the same subset in $\mathcal{C}$, and if r is a variable, then r' is a distinguished element in $\mathcal{C}$.*

Transition Rules. Let $R = F_0 \cup B_0$ be a RGV system and $CR_0 = CF_0 \cup B_0$ be its constrained form, where CF_0 denotes the constrained versions of the F_0 rules. It is easy to see that the relations $\to_R^*$ and $\to_{CF_0/B_0 \cup B_0}^*$ are identical.

We saturate the set $CR = CR_0$ under the following transition rules. In the process, we could generate constrained rules of the form $c \to \alpha$ **if** $\mathcal{C}$, where α is height 0, called CB-rules. Initially there are no CB-rules, and hence $CB_0 = \emptyset$. Starting with $CR_0 = CF_0 \cup B_0 \cup CB_0$, we apply the following inference rules exhaustively.

Ordered Chaining: $$\frac{CR = CR_1 \cup \{c \to_B s, \;\; u[t] \to_{CF} d \textbf{ if } \mathcal{C}\}}{CR \cup \{u[c] \to d \textbf{ if } \mathcal{C}\sigma\}}$$

if σ is the most-general unifier of s and t, and t is not a variable.

Extension 1: $$\frac{CR}{CR \cup \{c_S \to_B c_{\{i_1 \ldots i_k\}}\}}$$

if S is non-empty and $S \subseteq \{i_1 \ldots i_k\}$ and $\{c_{\{i_1\}} \ldots c_{\{i_k\}}\}$ is B-joinable.

Extension 2: $$\frac{CR = CR_1 \cup \{c_{S_1} \to_B c_{S_1'},\ \ c_{S_2} \to_B c_{S_2'}\}}{CR \cup \{c_{S_1 \cup S_2} \to_B c_{S_1' \cup S_2'}\}}$$

if $\{c_{S_1'}, c_{S_2'}\}$ is B-joinable.

Extension 3: $$\frac{CR = CR_1 \cup \{c_{S_1} \to_B f c_{S_{11}} \ldots c_{S_{1m}},\ \ c_{S_2} \to_B f c_{S_{21}} \ldots c_{S_{2m}}\}}{CR \cup \{c_{S_1 \cup S_2} \to_B f c_{S_{11} \cup S_{21}} \ldots c_{S_{1m} \cup S_{2m}}\}}$$

if, for all i, the set $\{c_{S_{1i}}, c_{S_{2i}}\}$ is B-joinable.

Constraint Solving 1: $$\frac{CR = CR_1 \cup \{c \to_{CB} d \textbf{ if } \mathcal{C}\}}{CR \cup \{c \to_B d\}}$$

if $\mathcal{C}$ contains only B-joinable sets of constants.

Constraint Solving 2: $$\frac{CR = CR_1 \cup \{c \to_{CB} x \textbf{ if } \mathcal{C}\}}{CR \cup \{c \to_B c_{S_1 \cup \ldots \cup S_k}\}}$$

if $\mathcal{C} = \mathcal{C}' \cup \{\{x; c_{S_1}, \ldots, c_{S_k}\}\}$, where $\mathcal{C}'$ contains only B-joinable sets of constants, and $\{c_{S_1}, \ldots, c_{S_k}\}$ is B-joinable.

Chaining B-rules 1: $$\frac{CR = CR_1 \cup \{c \to_B d,\ \ d \to_B e\}}{CR \cup \{c \to_B e\}}$$

Chaining B-rules 2: $$\frac{CR = CR_1 \cup \{c \to_B d,\ \ d \to_B f d_1 \ldots d_m\}}{CR \cup \{c \to_B f d_1 \ldots d_m\}}$$

The ordered chaining rule attempts to remove "peaks" from rewrite proofs, the extension rules add rules over the new constants that would be part of $\overline{R}$, the constraint solving rules simplify the constraints, and the last chaining rules add new B-rules so that all derivations using the final set of B-rules can be made increasing (defined below). Note that the rewrite rule $(u[c] \to d \textbf{ if } \mathcal{C}\sigma)$ introduced by ordered chaining can be a CF-, B-, or a CB-rule, but (i) the depth of $u[c]$ is bounded by the maximum depth of terms in $CF_0 \cup B_0$, (ii) the constraint $\mathcal{C}$ only contains depth zero terms, and (iii) every variable in $\mathcal{C}$ also occurs in either $u[c]$ or d. These observations together imply that saturation under the above inference rules can only generate finitely many different constrained rewrite rules (up to variable renaming). This implies that the inference rules can be implemented so as to guarantee termination.

Example 4. Let $R = \{0+x \to x,\ -a \to 0+a\}$. After flattening (where a new constant c_1 is introduced), moving nonlinearity into the constraints, and renaming constants we get $CF_0 = \{c_{\{0\}} + x_1 \to x_2 \textbf{ if } \{\{x_2; x_1\}\}, -c_{\{a\}} \to c_{\{1\}}\}$ and $B_0 = \{c_{\{1\}} \to c_{\{0\}} + c_{\{a\}}\}$. We have written $-c_{\{a\}}$ and $c_{\{0\}} + c_{\{a\}}$ instead of $f(c_{\{a\}}, -, -)$ and $f(c_{\{0\}}, c_{\{a\}}, +)$ for readability. Saturating the set $CF_0 \cup B_0$ under the above transition rules, we add the following rewrite rules:

$c_{\{1\}} \to x_2 \textbf{ if } \{\{x_2; c_{\{a\}}\}\}$	Ordered Chaining
$c_{\{1\}} \to c_{\{a\}}$	Constraint Solving 2
$-c_{\{1\}} \to c_{\{1\}}$	Ordered Chaining
$-c_{\{1\}} \to c_{\{a\}}$	Ordered Chaining
$c_{\{a\}} \to c_{\{a,1\}}$	Extension 1
$c_{\{1\}} \to c_{\{a,1\}}$	Extension 1
$c_{\{a,1\}} \to c_{\{a\}}$	Extension 2

We note that the transition rules do not add any more new rewrite rules, and the saturated set of rules is $CF_\infty = CF_0 \cup \{-c_{\{1\}} \to c_{\{1\}}, -c_{\{1\}} \to c_{\{a\}}\}$, $CB_\infty = \{c_{\{1\}} \to x_2 \textbf{ if } \{\{x_2; c_{\{a\}}\}\}\}$ and $B_\infty = B_0 \cup \{c_{\{1\}} \to c_{\{a\}}, c_{\{a\}} \to c_{\{a,1\}}, c_{\{1\}} \to c_{\{a,1\}}, c_{\{a,1\}} \to c_{\{a\}}\}$.

Lemma 3. *Let $CR_\infty = CF_\infty \cup B_\infty \cup CB_\infty$ denote the result of saturating the initial set CR_0, obtained from the set $R = F_0 \cup B_0$ (as described before). Then, $\to_R^* \subseteq \to^*_{(CF_\infty \cup CB_\infty)/B_\infty \,\cup\, B_\infty} \subseteq \overline{R}$.*

Example 5. Let R be as defined in Example 4. Add the rewrite rule $-x + x \to x$ to R. We will notice that the set $CF'_\infty = CF_\infty \cup \{-x_1 + x_2 \to x_3 \textbf{ if } \{\{x_3; x_1, x_2\}\}\}$ is a saturation of the resulting initial system. In the new initial system, the term $c_{\{a\}}$ is reachable from the term $- - c_{\{a\}} + c_{\{a\}}$ since

$$- - c_{\{a\}} + c_{\{a\}} \to -c_{\{1\}} + c_{\{a\}} \to -(c_{\{0\}} + c_{\{a\}}) + c_{\{a\}} \to -c_{\{a\}} + c_{\{a\}} \to c_{\{a\}}.$$

Using the saturated set,

$$- - c_{\{a\}} + c_{\{a\}} \to_{CF_\infty/B_\infty} -c_{\{1\}} + c_{\{a\}} \to_{B_\infty} -c_{\{a\}} + c_{\{a\}} \to_{CF_\infty/B_\infty} c_{\{a\}}.$$

A derivation $s \to_R^* t$ is said to be *increasing* [4] if whenever this derivation can be written as $s \to_R^* \circ \to_{R,p_1} \circ \to_R^* \circ \to_{R,p_2} \circ \to_R^* t$, then the position p_2 is not a prefix of p_1.

Lemma 4. *If $s \to_{CB_\infty/B_\infty} t$, then $s \to^*_{B_\infty} t$.*

Lemma 4 implies that the relation $\to^*_{(CF_\infty \cup CB_\infty)/B_\infty \,\cup\, B_\infty}$ coincides with the relation $\to^*_{CF_\infty/B_\infty \,\cup\, B_\infty}$. It would be possible to prove that this relation coincides with $\to^*_{CF_\infty/B_\infty} \circ \to^*_{B_\infty}$, but this does not imply that it is a rewrite closure presentation yet, since rewriting with $\to^*_{CF_\infty/B_\infty}$ can make terms bigger (if right-hand side is a variable, it can be instantiated by an arbitrarily large term). For this reason, we need to show that rewriting with $\to^*_{CF_\infty/B_\infty}$ can be restricted to size-decreasing rules while preserving the global relation $\to^*_{CF_\infty/B_\infty \,\cup\, B_\infty}$.

Definition 6. *Given a set C of constrained rules and a set D of rules, we write $s = u[l\sigma] \to_{C\surd D} u[r\sigma] = t$ if $(l \to r \textbf{ if } \mathcal{C})$ is a constrained rule in C and σ is a D-solution of $\mathcal{C}$ such that, if $\{x; \alpha_1, \ldots, \alpha_k\} \in \mathcal{C}$, then $Pos(x\sigma) = Pos(\alpha_1\sigma) \cup \ldots \cup Pos(\alpha_k\sigma)$.*

The new constants introduced in Section 4 are crucial for the following technical result.

Lemma 5. *Let $T = \{t_1 \ldots t_k\}$ be a set of B_∞-joinable terms. Then, there exists an effectively computable set of terms $S = \{s_1 \ldots s_l\}$ such that $Pos(s_1) = \ldots = Pos(s_l) = \bigcup_{i \in \{1,\ldots,k\}}(Pos(t_i))$ and $Reach_{B_\infty}(\{t_1, \ldots, t_k\}) = \bigcup_{i \in \{1,\ldots,l\}} Reach_{B_\infty}(s_i)$.*

Lemma 5 allows us to restrict the rewrite relation $\rightarrow^*_{CF_\infty/B_\infty}$ to the smaller relation of Definition 6.

Corollary 3. $\rightarrow^*_{CF_\infty/B_\infty \,\cup\, B_\infty} = \rightarrow^*_{CF_\infty\sqrt{}B_\infty \cup B_\infty}$

Henceforth, we fix $\succ$ to denote some reduction ordering that is an extension of the lexicographic combination of the height and size ordering on terms. Any such ordering $\succ$ will possess the subterm property, that is, $s \succ t$ if t is a proper subterm of s. We note that $\succ$ orients all the rules in $CF\sqrt{}B$ and $B^- \subseteq (\succ \cup \Sigma_0 \times \Sigma_0)$. The following proposition is established using proof simplification arguments [1].

Theorem 1. $\rightarrow^*_{CF_\infty\sqrt{}B_\infty \,\cup\, B_\infty} = \rightarrow^*_{CF_\infty\sqrt{}B_\infty} \circ \rightarrow^*_{B_\infty} = \overline{R}$*, and hence, the pair $(CF_\infty\sqrt{}B_\infty, B_\infty)$ is a rewrite closure for $\overline{R}$ w.r.t. the ordering $\succ$.*

Example 6. Consider the following derivation from Example 5:

$$- - c_{\{a\}} + c_{\{a\}} \rightarrow_{CF_\infty/B_\infty} -c_{\{1\}} + c_{\{a\}} \rightarrow_{B_\infty} -c_{\{a\}} + c_{\{a\}} \rightarrow_{CF_\infty/B_\infty} c_{\{a\}}.$$

The peak in the above proof can be eliminated by including the B_∞ rewrite step in the constrained rewriting step to get

$$- - c_{\{a\}} + c_{\{a\}} \rightarrow_{CF_\infty/B_\infty} -c_{\{1\}} + c_{\{a\}} \rightarrow_{CF_\infty/B_\infty} c_{\{a\}},$$

since $c_{\{1\}}$ and $c_{\{a\}}$ are B_∞-joinable.

Theorem 2. *Reachability and joinability of RGV rewrite systems is decidable.*

Proof. Using item 1 of Corollary 1 and item 4 of Corollary 2, deciding reachability and joinability for a RGV system R reduces to deciding reachability and joinability for the corresponding $\overline{R}$.

Theorem 1 establishes the decidability of reachability and joinability for $\overline{R}$ as follows. Given two terms s and t, the reachability problem $s \rightarrow^*_{CF_\infty\sqrt{}B_\infty \,\cup\, B_\infty} t$ can be decided by simply obtaining all terms reachable from s by $CF_\infty\sqrt{}B_\infty$, and checking whether one of them reaches t by rewriting with B_∞. The joinability problem $s \downarrow^*_{CF_\infty\sqrt{}B_\infty \,\cup\, B_\infty} t$ can be decided by obtaining all terms $s_1, \ldots, s_k$ and $t_1, \ldots, t_l$ that are reachable, from s and t respectively, by $CF_\infty\sqrt{}B_\infty$, and checking whether some s_i and t_j are B_∞-joinable.

7 Deciding Termination

For this section and the following one we make an additional assumption on the initial R: for every rule of the form $l \to x$, the variable x occurs in l. Note that, otherwise, R is trivially not terminating; and moreover, it is not confluent, since by our initial assumptions there is no rule of the form $x \to t$, and hence the equivalent terms y and z can not be joined.

Definition 7. *Let G be a term rewrite system. A term t is* context-reachable *from s by G if $s \to_G^+ c[t]_p$ for some context $c[\_]_p$. If the position p is not λ, then t is said to be* strictly context-reachable *from s.*

Lemma 6 characterizes nontermination of RGV systems R as described above.

Lemma 6. *R is nonterminating iff there exists a constant context-reachable from itself.*

The decidability of termination for ground rewrite systems [13] can now be generalized.

Theorem 3. *Termination of a RGV system R is decidable.*

Proof. On the one side, we can use $\to_{B_\infty}^*$ to check if a constant of the form $c_{\{i\}}$ is strictly context-reachable from itself by $\overline{R}$: this property is easy to check for ground rewrite systems like B_∞, for example by using tree automata techniques (see [3]). On the other, we can check if $c_{\{i\}} \to_R^+ c_{\{i\}}$ by considering all the terms t such that $c_{\{i\}} \to_R t$ (only one step with R), and checking if $t\overline{R}c_{\{i\}}$.

8 Deciding Confluence

Our goal is to decide the confluence of a RGV system R, but we show that this is equivalent to checking the confluence of $\overline{R}$. This, in turn, means we can just focus on the rewrite closure presentation of $\overline{R}$.

Lemma 7. *$\overline{R}$ is confluent iff R is confluent.*

From now on, we denote the sets CF_∞ and B_∞ by CF and B respectively. Moreover, we view the rewrite closure construction procedure as a black box and, for example, refer to the set of constants appearing in $CF \cup B$ as $\{c_1, c_2, \ldots\}$.

We will use the following result from [8].

Theorem 4. *Let $F \cup B$ be a rewrite closure of R with respect to the reduction ordering $\succ$ and let B_{CC} be a convergent presentation of B with respect to the same ordering. The rewrite system R is confluent if and only if the following two conditions are true:*
(a) The rewrite system $F \cup B_{CC}$ is locally confluent.
(b) $\downarrow_{F\cup B_{CC}} \subseteq \downarrow_{F\cup B}$.

In fact, the proof of decidability of confluence follows the scheme of [8], and the following subsections contain an adequate adaptation (and in some cases a simplification) of the proofs given in [8] for proving the decidability of the confluence of right-ground systems.

8.1 Canonical Instances of Constrained Rules

Note that $CF\surd B \ \cup \ B$ is a rewrite closure. We wish to use Theorem 4 to decide confluence of $CF\surd B \ \cup \ B$. In particular, we wish to check the (local) confluence of the terminating system $CF\surd B \ \cup \ B_{CC}$. We note here that, in the present case, the set B_{CC} is an *abstract congruence closure* for B^-, as defined in [2], and contains flat rules of the form $fc_1 \ldots c_m \to c$ and $c \to d$. The problem, however, is that $CF\surd B$ possibly represents infinitely many unconstrained rewrite rules. We solve this problem by considering a finite unconstrained rewrite system CCF that contains only certain *canonical* instances of CF; and we will show that checking the (local) confluence of $CCF \cup B_{CC}$ is enough. For the correct construction of CCF, we remove the useless rules of CF, i.e. the rules with unsatisfiable constraints.

Deletion 1:
$$\frac{CR \cup \{s \to t \ \mathbf{if}\ \mathcal{C} \cup \{\{c_{S_1}, \ldots, c_{S_k}, x_1, \ldots, x_l\}\}\}}{CR}$$

if the constants $c_{S_1}, c_{S_2}, \ldots, c_{S_k}$ are not B-joinable.

Deletion 2:
$$\frac{CR \cup \{s \to t \ \mathbf{if}\ \mathcal{C} \cup \{\{x; c_{S_1}, \ldots, c_{S_k}, x_1, \ldots, x_l\}\}\}}{CR}$$

if the constants $c_{S_1}, c_{S_2}, \ldots, c_{S_k}$ are not B-joinable.

Note that the removal of rules with unsatisfiable constraints does not modify the relations $\to_{CF/B \cup B}$ and $\to_{CF\surd B \cup B}$.

We define the *canonical solution* σ of a constraint $\mathcal{C}$ as the result of the following computation: Initially, let σ be the identity substitution. First, each constant in $\mathcal{C}$ is normalized by B_{CC}. Thanks to the deletion rules, there is no more than one distinct constant in every set in $\mathcal{C}$. We apply the following two rules exhaustively:
1. If $\{x_1, \ldots, x_k, c\}$ or $\{x_1; x_2, \ldots, x_k, c\}$ is in $\mathcal{C}$, then set $x_i\sigma \mapsto c$ for all i, and remove this constraint from $\mathcal{C}$.
2. If $\{x_1, \ldots, x_k\}$ or $\{x_1; x_2, \ldots, x_k\}$ is in $\mathcal{C}$, then set $x_2\sigma \mapsto x_1, x_3\sigma \mapsto x_1, \ldots, x_k\sigma \mapsto x_1$ and remove this constraint set from $\mathcal{C}$.
The substitution σ thus obtained is the *canonical solution* of the constraint $\mathcal{C}$.

We define the set

$$CCF = \{l\sigma \to r\sigma : l \to r \ \mathbf{if}\ \mathcal{C} \in CF \text{ and } \sigma \text{ is a canonical solution of } \mathcal{C}\}.$$

Proposition 1. *If $s \to_{CF\surd B} t$, then there exists t' such that $s \to^*_{CCF \cup B_{CC}} t' \leftarrow^*_{B_{CC}} t$. In fact, the relation $\leftrightarrow^*_{CF\surd B \cup B_{CC}}$ is identical to the relation $\leftrightarrow^*_{CCF \cup B_{CC}}$.*

Example 7. Consider $R = \{x + (-x) \to 0, x + 0 \to x, 0 \to -0\}$. The rewrite closure procedure of Section 6 starting from R gives $CF = \{x_1 + (-x_2) \to 0 \ \mathbf{if}\ \{\{x_1, x_2\}\}$, $x_1 + 0 \to x_2 \ \mathbf{if}\ \{\{x_2; x_1\}\}$, $x_1 + 0 \to 0 \ \mathbf{if}\ \{\{x_1, 0\}\}\}$ and $B = \{0 \to -0\}$, where $CF\surd B \cup B$ is a rewrite closure. Then $B_{CC} = \{-0 \to 0\}$ and $CCF = \{x + (-x) \to 0,\ x + 0 \to x,\ 0 + 0 \to 0\}$.

We now use Theorem 4 for characterizing the confluence of $CF\sqrt{B} \cup B$ in terms of CCF.

Corollary 4. *The rewrite system $CF\sqrt{B} \;\cup\; B$ is confluent if and only if the following two conditions are true:*
(i) $CCF \cup B_{CC}$ is (locally) confluent,[2]
(ii) $\downarrow_{CCF \cup B_{CC}} \subseteq \downarrow_{CF\sqrt{B} \,\cup\, B}$.

Condition (i) of Corollary 4 has crucial consequences: if $CF/B \;\cup\; B$ is confluent, then $CCF \cup B_{CC}$ is a finite and convergent presentation of the theory $CF/B \;\cup\; B$ and hence, it decides the word problem for $CF/B \;\cup\; B$.

This fact can be used to decide the unification problem for confluent RGV systems (this result has been recently obtained in [11]). Given two terms s and t and a substitution σ, $s\sigma \leftrightarrow^*_{CF\sqrt{B} \,\cup\, B} t\sigma$ iff $s\sigma \downarrow_{CCF \cup B_{CC}} t\sigma$, that is, the unification problem for a confluent right ground system reduces to the unification problem for a convergent system. Now, note that the rewrite system $CCF \cup B_{CC}$ is decreasing (its rules $l \to r$ are oriented by the ordering $\succ$ that cointains the proper subterm ordering), and hence unification modulo $CCF \cup B_{CC}$ is decidable [10].

Corollary 5. *The word problem and the unification problem are decidable for confluent RGV term rewrite systems.*

Now, deciding confluence reduces to checking conditions (i) and (ii). The rewrite system $CCF \cup B_{CC}$ is terminating; hence, checking condition (i) is decidable, and thus, the problem now reduces to checking condition (ii) of Corollary 4.

Example 8. Continuing the previous example, we notice that the rewrite system $CCF \cup B_{CC} = \{x + (-x) \to 0,\; x + 0 \to x,\; 0 + 0 \to 0,\; -0 \to 0\}$ is locally confluent. Hence to prove confluence of $CF\sqrt{B} \cup B$, we only need to test for the inclusion $\leftrightarrow^*_{CCF \cup B_{CC}} \subseteq \downarrow_{CF\sqrt{B} \,\cup\, B}$.

8.2 Top Stable Structures

Henceforth, we assume that $CCF \cup B_{CC}$ is confluent. We consider the problem of deciding if the equivalence relation $\leftrightarrow^*_{CCF \cup B_{CC}}$ is included in the joinability relation $\downarrow_{CF\sqrt{B} \cup B}$. We use an extension of the concept of top-stability [4] for checking this. For simplicity of proofs, we will use the interreduced (or fully-reduced) version $IRFB$ of $CCF \cup B_{CC}$. We define the interreduced version $IR(R)$ of R as follows:

$$IR(R) = \{l \to r : \exists l \to r' \in R. r' \to^!_R r \text{ and all proper subterms of } l \text{ are } R\text{-irreducible}\}$$

Proposition 2. *[8] If R is a convergent term rewriting system, then $R' = IR(R)$ is also a convergent presentation of R.*

Define $IRFB$ as $IR(CCF \cup B_{CC})$. Proposition 2 implies that $IRFB$ is a convergent presentation of $CCF \cup B_{CC}$.

[2] Note that $CCF \cup B_{CC}$ is terminating.

Example 9. In our running example, the set $CCF \cup B_{CC}$, as defined in Example 8, gives $IRFB = \{x + -x \to 0,\ x + 0 \to x,\ 0 + 0 \to 0,\ -0 \to 0\}$.

Let $IRFBG$ be the ground rules of $IRFB$. Formally, for every height 0 term α, we define $TopStable(\alpha)$ as

$$TopStable(\alpha) = \begin{cases} TopStable(NF_{IRFB}(\alpha)) & \textbf{if } \alpha \neq NF_{IRFB}(\alpha) \\ \{t = ft_1 \ldots t_m : t \to \alpha \in IRFB, \exists t^{\uparrow} : t^{\uparrow} \to^{*,nr}_{IRFBG} t \text{ and} & \\ t^{\uparrow} \text{ is } CF/(CCF \cup B_{CC})\text{-irreducible}\} & \text{otherwise.} \end{cases}$$

Lemma 8. *The equivalence relation induced by $CCF \cup B_{CC}$ is included in the joinability relation induced by $CF\sqrt{B}\ \cup\ B$ iff*
(i) constants $\{c_1, \ldots, c_n\}$ that are equivalent modulo $CCF \cup B_{CC}$ are B-joinable and
(ii) for every height zero term α, it is the case that (a) the set $TopStable(\alpha)$ has at most one term, and (b) if $t = ft_1 \ldots t_m \in TopStable(\alpha)$, then α is a constant, t is ground, and $\alpha \to^*_B fc_1 \ldots c_m$ for some constants $c_1, \ldots, c_m$ such that c_i and t_i are B_{CC}-equivalent.

8.3 Computing Top Stable Structures

The sets $TopStable(\alpha)$ are computed using a fixpoint computation that starts with $TS_0(c)$ and generates $TS_{i+1}(\alpha)$ by gradually adding more elements to $TS_i(\alpha)$. If $(t, t^{\uparrow}) \in TS_i(\alpha)$, it means that $t \in TopStable(\alpha)$ and $t^{\uparrow}$ is a witness for t. This construction is completely analogous to the one presented in [8].

Definition 8. *Two terms s and t are D*-similar*, denoted by $s =_D t$, if $Pos_{\leq D}(s) = Pos_{\leq D}(t)$ and for all $p \in Pos_{\leq D}(s)$, the root symbol of $s|_p$ is equal to the root symbol of $t|_p$.*

Formally, the fixpoint computation is defined as follows.

$$\begin{aligned} TS_0(\alpha) &= \{\ (l, l) :\ l \to \alpha \in IRFB,\ \text{ and } l \text{ is } CF/(CCF \cup B_{CC})\text{-irreducible}\} \\ TS_{j+1}(\alpha) &= \{\ (l, l') : l = l[c_1, \ldots, c_k] \to \alpha \in IRFB, \\ &\qquad l' = l[l'_1, \ldots, l'_k] \text{ where } (\_, l'_i) \in TS_j(c_i), \\ &\qquad l' \text{ is } CF/(CCF \cup B_{CC})\text{-irreducible and} \\ &\qquad \text{there is no } (l, l'') \in TS_j \text{ with } l' =_D l''\} \end{aligned}$$

The fixpoint computation is *prematurely terminated* if condition (ii) of Lemma 8 is ever violated for the sets $TS_i(\alpha)$. We denote by $TS_\infty(\alpha)$ the result of the fixpoint computation above in the case when it does not prematurely terminate. In this case, it is clear that for every pair $(t, t^{\uparrow}) \in TS_\infty(\alpha)$, $t \in TopStable(\alpha)$, that is, the fixpoint computation is sound. Lemma 9 shows that it is complete too.

Example 10. In our running example $IRFB = \{x + -x \to 0, x + 0 \to x, 0 + 0 \to 0, -0 \to 0\}$. The term $x+-x$ is reducible by $x_1+-x_2 \to 0$ **if** $\{\{x_1, x_2\}\}$, the term $x+0$ is reducible by $x_1+0 \to x_2$ **if** $\{\{x_2; x_1\}\}$, and the term $0+0$ is reducible by $x_1+0 \to 0$ **if** $\{\{x_1, 0\}\}$, but the term -0 is not. Hence, $TS_0(0) = \{(-0, -0)\}$. When computing $TS_1(x)$, we consider the rule $x + 0 \to x$ and the replacement of the 0 in its left-hand-side by -0. This gives the term $x + (-0)$, which is $CF/CCF \cup B_{CC}$ irreducible. Hence, the procedure prematurely terminates and $CF\sqrt{B} \cup B$ is not confluent.

Lemma 9. *If the fixpoint computation above does not prematurely terminate, then, for all α and all $t \in TopStable(\alpha)$ and every witness $t^{\uparrow}$ for t, there exists a witness $t^{\uparrow'}$ for t such that $(t, t^{\uparrow'}) \in TS_{\infty}(\alpha)$ and $t^{\uparrow} \leftrightarrow_D t^{\uparrow'}$.*

8.4 Deciding Confluence

Theorem 5. *The confluence of RGV rewrite systems is decidable.*

Proof. Given a RGV rewrite system R, a rewrite closure $CR = CF/B \ \cup \ B$ for $\overline{R}$ is constructed using the transformations and transition rules introduced in Section 6 (Theorem 1). The rewrite system R is confluent iff CR is confluent. An abstract congruence closure B_{CC} of B is then computed [2].

We compute CCF from CF and B_{CC} (Section 8.1) and Corollary 4 reduces the confluence problem to checking (i) local confluence of $CCF \cup B_{CC}$, which is decidable, and (ii) the inclusion $\downarrow_{CCF \cup B_{CC}} \subseteq \downarrow_{CF\surd B \, \cup \, B}$. Lemma 8 reduces checking condition (ii) to verifying two conditions. The first of these two is easily checked since B is ground (for example, see [3]). For the second, we note that the relations $\rightarrow_B^*$ and $\leftrightarrow_{B_{CC}}^*$ are decidable since B and B_{CC} are ground rewrite systems. Hence, it is sufficient to show computability of the sets $TopStable(\alpha)$ and this is established by Lemma 9.

9 Complexity

Termination and confluence of *shallow* RGV systems can be decided in $2^{O(n)}$ time assuming that the maximum arity m of a function symbol is a constant. This upper-bound is tight since these problems can be shown to be EXPTIME-hard by reductions similar to the ones presented in [6]. The time complexity is obtained by bounding the size of the rewrite closure. Note that the total number of possible rules is now $(2^n)^{2m} = 2^{2mn}$ and its computation has cost $(2^{2mn})^4 = 2^{8mn}$. For termination, a fixpoint computation algorithm can decide if constants are strictly context reachable from themselves by B with cost $(2^{2mn})^4 = 2^{8mn}$, and deciding reachability of some height 1 terms into constants has cost $(2^{2mn})^4 = 2^{8mn}$. For confluence, the number of $CCF \cup B_{CC}$ rules is 2^{2mn}, its computation has cost $(2^{2mn})^2 = 2^{4mn}$, and checking its confluence has cost $(2^{2mn})^3$. In the computation of topstable structures, the $TS_i(c)$ contain terms of height at most 2, but two introduced terms must be different in positions smaller than or equal to 1; hence, the $|TS_i(c)|$'s are bounded by 2^{mn}, and the total cost for computing them is $(2^{2mn})^4 = 2^{8mn}$. If we drop the *shallowness* assumption, then we get a $2^{O(n^2)}$ complexity for deciding these properties (assuming m is a constant).

10 Conclusion

We have presented new decidability results for several fundamental properties of RGV systems, such as reachability, joinability, termination, and confluence. These results are interesting from many different perspectives. First, since our approach is based on extending and generalizing existing concepts that have proven useful for other simpler classes of term rewriting systems, we clearly bring out what additional technical issues are presented by richer classes of term rewriting systems. Second, when specialized to ground systems, there is a correspondence between our approach based on rewrite closure and the tree transducer techniques. However, using our approach, we have been successful in lifting the decidability results to several subclasses of nonground systems. Third, these new decidability results lie at the fringes of the decidability space as several properties are undecidable for term rewrite systems with slightly different restrictions. For future work, it would be interesting to study if these techniques can be adapted to incorporate axioms that can not be expressed by RGV rules.

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