# Termination of Rewrite Systems with Shallow Right-Linear, Collapsing, and Right-Ground Rules*

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**Abstract.** We show that termination is decidable for rewrite systems that contain shallow and right-linear rules, collapsing rules, and right-ground rules. This class of rewrite systems is expressive enough to include interesting rules. Our proof uses the fact that this class of rewrite systems is known to be regularity-preserving and hence the reachability and joinability problems are decidable. Decidability of termination is obtained by analyzing the nonterminating derivations.

## 1 Introduction

Term rewriting systems are Turing-complete models of computation that specify rules for replacing certain patterns in terms by equivalent, in some cases simpler, other terms. Simpler models of computation result by imposing additional constraints on the form of terms in a rewrite system. For instance, if variables are not allowed, we get *ground* term rewrite system, which have been extensively studied, mainly via mapping them to tree automata [2]. More complex models of computation arise by allowing restricted variable occurrences in the term rewrite system (or the tree automata transitions).

Termination is one of the central properties of rewrite systems. Termination guarantees that any expression cannot be infinitely rewritten, and hence, the existence of a normal form for it. As we go from simple to more general classes of rewrite systems, the complexity of deciding termination increases until it becomes undecidable. For example, while termination is decidable in polynomial time for ground term rewriting systems [13, 16], it is undecidable for general rewrite systems and string rewrite systems [13]. It is, therefore, fruitful to identify the decidability barrier and study decidability issues for some intermediate classes, especially if these classes are expressive enough to capture interesting rules.

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There are several negative, and few positive, results on decidability of termination for classes of rewrite system. Termination is undecidable for even one (non-linear) rule [4]. Termination is usually established using well-founded orderings [6]. It is undecidable whether a single term rewriting rule can be proved terminating using a simplification ordering [15] or a monotonic ordering total on ground terms [7]. On the other hand, several powerful techniques and implementations exist that can automatically prove termination of many rewriting systems [8, 3, 12, 1]. These systems are based on combinations of techniques such as the use of well-founded term orderings, transformations, semantic interpretations, and dependency-pairs. The success of these tools suggests the natural question: is there an interesting and large class of rewrite systems for which termination is indeed decidable?

In this context, we consider term rewriting systems that contain shallow right-linear rules, collapsing rules, and right-ground rules. In a shallow right-linear rule $l \to r$, every variable occurs at most once in $r$, and all variables in $l, r$ occur at depth 0 or 1. Some examples of shallow right-linear rules are $0 \wedge x \to 0$, $x \wedge x \to x$, $1 \wedge x \to x$, $1 \vee x \to 1$, $x \vee x \to x$, $0 \wedge x \to x$, $x \wedge y \to y \wedge x$ and $x \vee y \to y \vee x$. A rule of the form $l \to x$, where $x$ is a variable, is called collapsing. For example, $\neg(\neg x) \to x$ is a collapsing rule. A rule $l \to r$, where $r$ is a ground term, is a right-ground rule. For example, $x \wedge (\neg x) \to 0$, $x \vee (\neg x) \to 1$ are right-ground rules.

Our proof of decidability of termination relies on the decidability of reachability and joinability. Takai, Kaji, and Seki [17] showed that right-linear finite-path-overlapping systems effectively preserve recognizability. The class of rewrite systems defined by shallow right-linear, collapsing, and right-ground rules, is right-linear and finite-path-overlapping, and hence it follows that the reachability and joinability problems for this class is decidable. We point out here that reachability is known to be undecidable for linear TRS's, and also for shallow TRS's [14].

In this paper, we prove the decidability of termination for TRS's that contain shallow right-linear, collapsing, or right-ground rules. We use the decidability of reachability and joinability for this class as a black box. For termination, we give a checkable characterization based on some reachability conditions and the termination of a restricted rewrite system related with the original one.

In Section 2 we introduce some basic notions and notations. In Section 3 we present termination-preserving transformations that replace the shallow right-linear rules by flat right-linear rules and that replace the right-ground rules by right-constant rules. This section is quite easy and similar to parts of other previous works, but not identical, and allows us to simplify the arguments in the rest of the paper. In Section 4 we characterize the termination property for flat right-linear systems that contain additional collapsing and right-constant rules, and prove its decidability.

## 2 Preliminaries

We use standard notation from the term rewriting literature. A signature $\Sigma$ is a (finite) set of function symbols, which is partitioned as $\cup_i \Sigma_i$ such that $f \in \Sigma_n$ if the arity of $f$ is $n$. Symbols in $\Sigma_0$, called *constants*, are denoted by $a, b, c, d$, with possible subscripts. The elements of a set $\mathcal{V}$ of variable symbols are denoted by $x, y$ with possible subscripts. The set $\mathcal{T}(\Sigma, \mathcal{V})$ of *terms* over $\Sigma$ and $\mathcal{V}$, *position* $p$ in a term, *subterm* $t|_p$ of term $t$ at position $p$, and the term $t[s]_p$ obtained by replacing $t|_p$ by $s$ are defined in the standard way. For example, if $t$ is $f(a, g(b, h(c)), d)$, then $t|_{2.2.1} = c$, and $t[d]_{2.2} = f(a, g(b, d), d)$. We write $p_1 \succ p_2$ (or, $p_2 \prec p_1$) if $p_2$ is a proper prefix of $p_1$. By $Vars(t)$ we denote the set of all variables occurring in $t$. The *height* of a term $s$ is 0 if $s$ is a variable or a constant, and $1 + max_i height(s_i)$ if $s = f(s_1, \ldots, s_m)$. Usually we will denote a term $f(t_1, \ldots, t_n)$ by the simplified form $ft_1 \ldots t_n$, and $t[s]_p$ by $t[s]$ when $p$ is clear by the context or not important.

A substitution $\sigma$ is sometimes presented explicitly as $\{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$. We assume standard definitions for a *rewrite rule* $l \to r$, a *rewrite system* $R$, the *one step rewrite relation at position* $p$ *induced by* $R$ $\to_{R,p}$, and the *one step rewrite relation induced by* $R$ (at any position) $\to_R$. If $p = \lambda$, then the rewrite step $\to_{R,p}$ is said to be applied *at the topmost position* (at the root) and is denoted by $s \to_R^r t$; it is denoted by $s \to_R^{nr} t$ otherwise.

The notations $\leftrightarrow$, $\to^+$, and $\to^*$, are standard. $R$ is terminating if no infinite derivation $s_1 \to_R s_2 \to \cdots$ exists. A term $t$ is *reachable* from $s$ by $R$ (or, $R$-reachable) if $s \to_R^* t$. A term $s$ is *R-irreducible* (or, in $R$-normal form) if there is no term $t$ such that $s \to_R t$. We denote by $s \to^! t$ the fact that an irreducible term $t$ is reachable from $s$ by the $\to$ relation. If $s$ is a term and $S$ is a set of terms, then we define $Reach(s) = \{t : s \to_R^* t\}$ and $Reach(S) = \bigcap_{s \in S} Reach(s)$. A set $S$ of two or more terms is *R-joinable* if $Reach(S) \neq \emptyset$. A *(rewrite) derivation or proof* (from $s$) is a sequence of rewrite steps (starting from $s$), that is, a sequence $s \to_R s_1 \to_R s_2 \to_R \ldots$.

A term $t$ is called *ground* if $t$ contains no variables. It is called *shallow* if all variable positions in $t$ are at depth 0 or 1. It is called linear if every variable occurs at most once in $t$. A rule $l \to r \in R$ is called *right-ground* if $r$ is ground, and *collapsing* if $r$ is a variable. It is called *shallow right-linear* if the term $r$ is linear, and both $l, r$ are shallow, and *flat* if both $l, r$ are height 0 or 1 terms.

## 3 Termination-Preserving Transformations

Let $R$ be such that for every rule $l \to r \in R$, either $r$ is ground, or $r$ is a variable, or $l \to r$ is shallow and right-linear. Henceforth, we also assume that in every rule $l \to r$ we have $Vars(r) \subseteq Vars(l)$ (this is usual for rewrite rules, and without this property the corresponding rewrite system is trivially non-terminating).

By replacing non-constant ground terms by new constants, as described formally by the following two transformation rules, we can transform the rewrite system $R$ into a rewrite system $R'$ such that for every rule $l \to r \in R'$, either $r$ is a constant, or $r$ is a variable, or $l \to r$ is flat and right-linear.

$$\frac{l \to r[s]}{l \to r[c], c \to s} \qquad \frac{l[s] \to r}{l[c] \to r, s \to c}$$ if $s$ is non-constant and ground; $c$ a new constant

We show in Lemma 3 that application of these two transformation rules is terminating, and hence we can exhaustively apply them. As an optimization we can replace multiple instances of $s$ on LHS (equivalently, RHS) by the *same* constant $c$, but we use the unoptimized version here to keep proofs simple. We prove below that applying these two transformation rules preserves termination. We remark here that this transformation is very similar (though not identical) to the one that preserves confluence, see [11].

**Lemma 1.** *Let $R'$ be obtained from $R$ using one of the two transformation rules described above. For every derivation $t_1 \to_R t_2 \to_R t_3 \to_R \cdots$, there exists a derivation $t_1 \to^+_{R'} t_2 \to^+_{R'} t_3 \to^+_{R'} \cdots$.*

*Proof.* Suppose $t_i \to_{l\to r,\sigma,p} t_{i+1}$, where $l \to r \in R$. If $l \to r$ is also present in $R'$, then clearly $t_i \to_{R'} t_{i+1}$. If not, then suppose $r|_q = s$ and $l \to r \in R$ is replaced by $l \to r[c]_q$ and $c \to s$ in $R'$. In this case, $t_i \to_{l\to r[c],\sigma,p} t_{i+1}[c]_{p.q} \to_{c\to s,id,p.q} t_{i+1}[s]_{p.q} = t_{i+1}$. In the other case, suppose $l|_q = s$ and $l \to r \in R$ is replaced by $l[c]_q \to r$ and $s \to c$ in $R'$. Now we have $t_i \to_{s\to c,id,p.q} t_i[c]_{p.q} \to_{l[c]\to r,\sigma,p} t_{i+1}$. This completes the proof. □

**Lemma 2.** *Let $R'$ be obtained from $R$ using one of the two transformation rules described above. Let $c$ be the new constant that names some non-constant ground term $s$. For every infinite derivation $t_1 \to_{R'} t_2 \to_{R'} \cdots$ over $\mathcal{T}(\Sigma \cup \{c\}, \mathcal{V})$, there exists an infinite derivation $t_1\sigma \to^*_R t_2\sigma \to^*_R \cdots$ over $\mathcal{T}(\Sigma, \mathcal{V})$, where $\sigma$ is $\{c \mapsto s\}$ and is applied as a substitution interpreting $c$ as a variable.*

*Proof.* Suppose $R' = (R - \{l[s] \to r\}) \cup \{l[c] \to r, s \to c\}$. Consider the step $t_i \to_{l'\to r',\rho,p} t_{i+1}$, where $l' \to r' \in R'$. There are three cases. (1) If $l' \to r' \in R$, then since the constant $c$ is new and not present in $R$, it follows that $t_i\sigma \to_{l'\to r',\rho\sigma,p} t_{i+1}\sigma$.(2) If $l' = l[c]$ and $r' = r$, then we can use the rewrite rule $l[s] \to r$ from $R$ to get $t_i\sigma \to_{l[s]\to r,\rho\sigma,p} t_{i+1}\sigma$.(3) If $l' = s$ and $r' = c$, then $t_i\sigma = t_{i+1}\sigma$ and there is no corresponding step in the $R$-derivation. Since $\{s \to c\}$ is terminating, case (1) and (2) happen infinitely often, and hence the derivation $t_1\sigma \to^*_R t_2\sigma \to^*_R \cdots$ is an infinite derivation. Finally, we complete the proof by saying that the argument for the case when $R' = (R - \{l \to r[s]\}) \cup \{l \to r[c], c \to s\}$ can be done similarly. □

Let $R \vdash R'$ denote that $R'$ is obtained from $R$ using an application of the either of the two transformation rules.

**Lemma 3.** *Every derivation $R_1 \vdash R_2 \vdash R_3 \vdash \cdots$ is necessarily finite. If $R_1$ is such that for every $l \to r \in R$, either $r$ is ground, or $r$ is a variable, or $l \to r$ is shallow and $r$ is linear, then the final rewrite system $R$ obtained as $R_1 \vdash^! R$ is such that for every $l \to r \in R$, either $r$ is a constant, or $r$ is a variable, or $l \to r$ is flat and $r$ is linear.*

*Proof.* Let $measure(R)$ be the multiset consisting of the depths of $l$ and $r$ for every $l \rightarrow r \in R$. If $R \vdash R'$, then $measure(R) >^m measure(R')$, where $>^m$ is the multiset extension of the regular greater-than $>$ ordering on the naturals. Hence the relation $\vdash$ is well-founded. If $R_1 \vdash^! R$, and $R$ violates the second claim, then one of the two transformation rules will be applicable on $R$, thus contradicting that $R$ is the normal form of $R$ w.r.t $\vdash$. □

The following theorem is now an easy consequence of Lemma 1, Lemma 2, and Lemma 3.

**Theorem 1.** *If $R$ is any collection of right-ground rules, right-variable rules, and shallow and right-linear rules, then $R$ can be transformed into $R'$ such that $R'$ is a collection of right-constant rules, right-variable rules, and flat and right-linear rules. Furthermore, $R$ is terminating if and only if $R'$ is terminating.*

Additionally, with a transformation identical to the one presented in [11], we can encode function symbols with nonzero arity using just one function symbol, say $f$, with sufficiently large arity $m$. Hence, we can assume that $\Sigma = \Sigma_0 \cup \{f\}$, where $f$ is of arity $m$. This encoding preserves termination and is done just to simplify the proofs.

## 4 Termination

As a consequence of Theorem 1, we can without loss of generality assume that all rules in $R$ are right-constant, right-variable, or flat and right-linear; and that $\Sigma$ contains only one non-constant function symbol $f$ of arity $m$.

Termination is decidable for right-ground term rewriting systems [5] and also for the more general class that also has right-variable rules [9]. An important idea used in [9] for handling nonlinear left-hand side terms is that of treating sets of constants (generalized to *terms* in this paper) as first-class objects (terms). Intuitively, a set $S \subseteq \mathcal{T}(\Sigma, \mathcal{V})$ of terms *represents* any (all) terms that are $R$-reachable from every term $s \in S$. For example, under this interpretation, the rewrite rule $fxx \rightarrow fax$ can rewrite $fS_1S_2$ to $fa(S_1 \cup S_2)$, *if* there is some term $R$-reachable from every term in $S_1 \cup S_2$. This is the basis for Definition 1.

A second observation we make in this paper is that the termination of $R$ can be decomposed into termination of right-constant or right-variable rules and the termination of flat and right-linear rules. In particular, this means that the rules $l \rightarrow r$ where $depth(l) = depth(r) = 1$, called *permutation rules*, play an important role in characterizing the termination of the rewrite system $R$.

**Definition 1.** *[UJoin, UPerm] Let $R$ be a flat right-linear TRS over $\Sigma$. Define an infinite set $K = \{S : \emptyset \neq S \subseteq \mathcal{T}(\Sigma, \mathcal{V})$ is $R$-joinable$\}$ of new constants (every set in $K$ represents a constant) called set-constants. The rewrite systems*

*$UJoin(R)$, $Perm(R)$, and $UPerm(R)$ (over $\Sigma \cup K$) are defined as follows:*

$$\begin{aligned}
UJoin(R) &= \{S \to \{c\} : S \in K,\ c \in \Sigma_0,\ c \in Reach(S),\ \textit{and } S \neq \{c\}\} \\
Perm(R) &= \{l \to r \in R : depth(l) = depth(r) = 1\} \\
UPerm(R) &= \{fS_1 \ldots S_m \to fT_1 \ldots T_m : \exists fs_1 \ldots s_m \to ft_1 \ldots t_m \in Perm(R), \\
&\qquad S_p \in K, T_p \in K \textit{ for all } p \in \{1, \ldots, m\} \textit{ AND} \\
&\qquad S_p = \{s_p\} \textit{ whenever } s_p \in \Sigma_0 \textit{ AND} \\
&\qquad T_p = \{t_p\} \textit{ whenever } t_p \in \Sigma_0 \textit{ AND} \\
&\qquad \forall x \in Vars(fs_1 \ldots s_m),\ \textstyle\bigcup_{s_i = x} S_i \textit{ is } R\textit{-joinable, and if some } t_j \textit{ is } x, \\
&\qquad \textit{then } T_j \textit{ is } \textstyle\bigcup_{s_i = x} S_i\}
\end{aligned}$$

Note that the set $K$ is infinite. The rewrite system $UPerm(R)$ is ground (though constants of the form $\{x\} \in K$ may appear in it). The sets $UPerm(R)$ and $UJoin(R)$ are theoretical constructions possibly containing infinitely many rules. The termination characterization of Lemma 6 applies the rewrite system $UPerm(R) \cup UJoin(R)$ only on terms $fS_1 \ldots S_m$ where each $S_i \subseteq \Sigma_0$. Hence, the relevant rules of $UPerm(R) \cup UJoin(R)$ are those that only contain set-constants $S$ s.t. $S \subseteq \Sigma_0$.

*Example 1.* If $R = \{fxxx \to fxc_1c_2,\ c_2 \to c_1\}$, then the set $UJoin(R)$ restricted to set-constants over $\Sigma_0$ is $\{\{c_2\} \to \{c_1\},\ \{c_1, c_2\} \to \{c_1\}\}$. The set $UPerm(R)$ restricted similarly contains the 27 rules: $\{fS_1S_2S_3 \to fS\{c_1\}\{c_2\} : \emptyset \neq S_1, S_2, S_3 \subseteq \{c_1, c_2\}, S = S_1 \cup S_2 \cup S_3\}$. The full set $UPerm(R)$ contains several other rules, for example, rules of the form $fSSS \to fSc_1c_2$, where $S$ is any subset of terms that are $R$-joinable.

The notion of set-constants and the new definition of rewriting (using *U*nions) induced by $R$, and captured by $UPerm(R)$, allows us to
(a) project certain infinite $R$-derivations onto infinite $(UPerm(R) \cup UJoin(R))$-derivations over ground terms (Lemma 4 and Example 2); and
(b) lift infinite $(UPerm(R) \cup UJoin(R))$-derivations starting from a flat ground term to an infinite $R$-derivation (Lemma 5 and Example 3).
The consequence of these two lemmas is that termination of $R$ restricted to derivations with only permutation steps at root positions is equivalent to termination with $(UPerm(R) \cup UJoin(R))$-derivations starting from flat ground terms.

### 4.1 Projecting Rewrite Derivations

Consider an infinite derivation $fcc(gb) \to_R fc(gb)b \to_R \cdots$ using some rewrite system $R$ containing the permutation rule $fxxy \to fxyb$. We will first project this derivation onto the $(UPerm(R) \cup UJoin(R))$-derivation $f\{c\}\{c\}\{gb\} \to f\{c\}\{gb\}\{b\} \to \cdots$, and thereafter remove the non-constant subterm $gb$ from it by carefully analyzing the role of $gb$ in the subsequent derivation.

**Lemma 4.** *Let $s_1 = fs_{11} \dots s_{1m} \to_R s_2 \to \cdots$ be an infinite R-derivation that contains infinitely many top steps, where all of them are with rules in $Perm(R)$. Then there is an infinite $(UPerm(R) \cup UJoin(R))$-derivation starting from a term of the form $fS_1 \dots S_m$, where every $S_i$ is included in $\Sigma_0$ and is R-joinable.*

*Proof.* We project the derivation $s_1 \to s_2 \cdots$ onto an infinite $(UPerm(R) \cup UJoin(R))$-derivation $t_1 \to^* t_2 \cdots$.

First, if $s_1$ is of the form $fs_{11} \dots s_{1m}$, then let $t_1$ be $f\{s_{11}\} \dots \{s_{1m}\}$. We inductively define $t_{i+1}$ as follows. If $s_i \to s_{i+1}$ is a non-root rewrite step, then $t_{i+1} = t_i$. If $s_i \to s_{i+1}$ is a root rewrite step using a permutation rule $l \to r$, then, we define $t_{i+1}|_p$ as $\{s_{i+1}|_p\}$ if $r|_p$ is a constant (and hence equal to $s_{i+1}|_p$), and as $\bigcup_{l|_{p'}=r|_p} t_i|_{p'}$ if $r|_p$ is a variable.

By construction, all $t_i$'s are flat terms where the depth 1 constants are sets $S$ that contain either constants of the original signature or the initial subterms $s_{11}, \dots, s_{1m}$, i.e., if $t_i|_p = S$, then $S \subseteq \{s_{11}, \dots, s_{1m}\} \cup \Sigma_0$.

We prove by induction on $i$ that every $s_i|_p$ is reachable from all terms in $t_i|_p$, that is, $s_i|_p \in Reach(t_i|_p)$ for all $p$. This is trivially true for $s_1$ and $t_1$. If $s_{i+1}$ is obtained from $s_i$ using a non-root rewrite step, then it follows that $t_i = t_{i+1}$ (by definition), $s_i|_p \in Reach(t_i|_p)$ (by induction hypothesis), and $s_i|_p \to s_{i+1}|_p$, which together implies that $s_{i+1}|_p \in Reach(t_{i+1}|_p)$. If $s_{i+1}$ is obtained from $s_i$ with a root rewrite step using a permutation rule $l \to r$, then, (i) for positions $p$ s.t. $r|_p \in \Sigma_0$ the result directly follows since for such $p$ we have $t_{i+1}|_p$ is $\{s_{i+1}|_p\}$, and (ii) for positions $p$ s.t. $r|_p \in \mathcal{V}$ we have $t_{i+1}|_p = \bigcup_{l|_{p'}=r|_p} t_i|_{p'}$ (by definition), each $s_i|_{p'} \in Reach(t_i|_{p'})$ (by induction hypothesis), and $s_{i+1}|_p$ coincides with all $s_i|_{p'}$ such that $l|_{p'} = r|_p$, which together implies that $s_{i+1}|_p \in Reach(t_{i+1}|_p)$.

Now we show that if $s_i \to s_{i+1}$ with a permutation rule $l \to r$, then $t_i \to^*_{UJoin(R)} \to_{UPerm(R)} t_{i+1}$, where the last step is done with the $UPerm(R)$ rule $fS_1 \dots S_m \to t_{i+1}$ constructed from $l \to r$ by setting $S_j = \{l|_j\}$ if $l|_j$ is a constant, and $S_j = t_i|_j$ is $l|_j$ is a variable. The term $t_i$ may differ from $fS_1 \dots S_m$ at positions $p$ such that $l|_p$ is a constant. For each such position $p$, $S_p$ coincides with $\{s_i|_p\}$, and by the previous fact, this $s_i|_j$ is reachable from all terms in $t_i|_p$, and hence, a $UJoin(R)$ rule $t_i|_p \to S_p$ exists. Hence we conclude that $t_i \to^*_{UJoin(R)} fS_1 \dots S_m \to_{UPerm(R)} t_{i+1}$.

Since the derivation $s_1 \to \dots$ contains infinite root rewrite steps with permutation rules, the derivation $t_1 \to \dots$ is also infinite. By right-linearity of $R$ and the definition of $UPerm(R)$ and $UJoin(R)$, the number of occurrences of the non-constant terms $\{s_{11}, \dots, s_{1m}\}$ in the set-constants $S$ cannot increase in the infinite derivation $t_1 \to \dots$. If some non-constant $s_{1j}$'s are persisting, then the sets in which they occur can only become larger. Choose $i$ large enough so that the sets containing non-constant terms do not change any more in the derivation $t_i \to t_{i+1} \to \cdots$. We can map this infinite derivation into a new one over flat terms, in which all sets contain only constants, by eliminating the non-constant $s_{1j}$ occurrences. Before doing it, we pick a fixed constant $c \in \Sigma_0$. Now, if $S$ contains some non-constant $s_{1j}$ and also some constants, then we just remove such $s_{1j}$ from the set $S$. If $S$ contains no constants, but only terms such as $s_{1j}$, then we replace $S$ by $\{c\}$. With these replacements, it is easily verified that the derivation $t_i \to t_{i+1} \to \cdots$ is transformed into a new infinite rewriting derivation $t'_i \to t'_{i+1} \to \cdots$ with all terms flat and all set-constants only containing constants from $\Sigma_0$. This completes the proof. $\square$

*Example 2.* Let $\Sigma = \{f, g, a, b, c\}$, where $arity(f) = 3$, $arity(g) = 1$, and arity of all other symbols is 0. Consider the rewrite system:

$$R = \{fxxy \to fxyb,\ fxyd \to fcxa,\ a \to gb,\ b \to d,\ gb \to c\}.$$

Note that $a$ normalizes to $c$ and $b$ normalizes to $d$. Consider the following infinite $R$-derivation obtained by successively normalizing the depth 1 subterms (denoted by superscript $*, nr$) and applying the appropriate permutation rule from $R$ (denoted by superscript $r_1, r_2$ for the two rules respectively).

$$\begin{aligned} fccgb &\to^{*,nr} fccc \to^{r_1} fccb \to^{*,nr} fccd \to^{r_1} fcdb \\ &\to^{*,nr} fcdd \to^{r_2} fcca \to^{*,nr} fccc \to^{r_1} \cdots \end{aligned}$$

We project this derivation in two steps. In the first step, we ignore the $nr$-steps and use the derived $UPerm(R)$ rule to perform the $r$-steps. Note that we need to use the $UJoin(R)$ rule $\{b\} \to \{d\}$ below. We get the following derivation:

$$\begin{array}{llll} f\{c\}\{c\}\{gb\} & \to^{r_1} f\{c\}\{gb\}\{b\} & \to^{r_1} f\{c, gb\}\{b\}\{b\} & \to^{nr} f\{c, gb\}\{b\}\{d\} \\ & \to^{r_2} f\{c\}\{c, gb\}\{a\} & \to^{r_1} f\{c, gb\}\{a\}\{b\} & \to^{r_1} f\{a, c, gb\}\{b\}\{b\} \\ & \to^{nr} f\{a, c, gb\}\{b\}\{d\} & \to^{r_2} f\{c\}\{a, c, gb\}\{a\} & \to^{r_1} f\{a, c, gb\}\{a\}\{b\} \\ & \to^{r_1} f\{a, c, gb\}\{b\}\{b\} & \to^{nr} \cdots & \end{array}$$

In the second step, we notice that the set-constants in the derivation starting from $f\{a, c, gb\}\{b\}\{b\}$ do not change, and hence, we forget the nonconstants in the sets and get the following $(UPerm(R) \cup UJoin(R))$-derivation starting from $f\{a, c\}\{b\}\{b\}$.

$$\begin{aligned} f\{a, c\}\{b\}\{b\} &\to^{nr} f\{a, c\}\{b\}\{d\} \to^{r_2} f\{c\}\{a, c\}\{a\} \to^{r_1} f\{a, c\}\{a\}\{b\} \\ &\to^{r_1} f\{a, c\}\{b\}\{b\} \to^{nr} \cdots \end{aligned}$$

This is the required nonterminating derivation.

### 4.2 Lifting Rewrite Derivations

We next prove the converse of Lemma 4 under the assumption that $UJoin(R)$ is terminating. First we need the notions of a position being *related to* and *going to* other positions in a permutation rule. For example, in the rule $fxxy \to fxyb$, position 1 is related to position 2 in the left-hand side term and positions 1 and 2 both go to position 1 on the right-hand side term. The goes-to relation is well-defined since rules are right-linear. We naturally generalize this to $UPerm(R)$-rules below.

**Definition 2.** *Let $fS_1 \ldots S_m \to fT_1 \ldots T_m$ be a rule in $UPerm(R)$, and let $fs_1 \ldots s_m \to ft_1 \ldots t_m$ be the rule of $R$ from which it is constructed. (Since a* *rule in $UPerm(R)$ can be constructed from different rules in $R$, we are assuming an implicit arbitrary selection).*

*We say that* $i_1, \ldots, i_k$ *are* positions related to $i$ *in* $fS_1 \ldots S_m \to fT_1 \ldots T_m$ *if* $s_i$ *is a variable and* $s_{i_1} = \ldots = s_{i_k} = s_i$. *We say that position* $i$ goes to *position* $j$ *in* $fS_1 \ldots S_m \to fT_1 \ldots T_m$ *if* $s_i$ *is a variable and* $t_j = s_i$. *We say that* $i$ *is an* original constant position *in* $fS_1 \ldots S_m \to fT_1 \ldots T_m$ *if* $s_i$ *is a constant.*

**Lemma 5.** *Suppose $UJoin(R)$ is terminating. Let $s_1 \to s_2 \to \cdots$ be an infinite $(UPerm(R) \cup UJoin(R))$-derivation, where $s_1 = fS_1 \ldots S_m$ and every $S_i \subseteq \Sigma_0$ is a set of $R$-joinable constants. Then, $R$ is nonterminating.*

*Proof.* We associate a sequence of positions $i_1, i_2, \ldots$ with every depth 1 position $i$ in $s_1$ as follows: $i_1 = i$ and for every $j \geq 1$, (a) $i_{j+1} = i_j$ if the rewrite rule used in $s_j \to s_{j+1}$ is from $UJoin(R)$, (b) $i_j$ goes to $i_{j+1}$ if the rewrite rule $s_j \to s_{j+1}$ is in $UPerm(R)$, and (c) $i_{j+1}$ is undefined (and the sequence terminates) if the position $i_j$ does not go to any position in the rule $s_j \to s_{j+1} \in UPerm(R)$. Note that this sequence is uniquely defined for every $i$ since $R$ is right-linear. Thus, the sequence associated with $i$ can be either finite or infinite. It is easy to prove inductively that, if $i_1 \ldots$ is the sequence associated with $i = i_1$ in $s_1$, then, for all $i_k$ in this sequence, the set $s_k|_{i_k}$ is $R$-joinable and $Reach(s_1|_{i_1}) \supseteq Reach(s_k|_{i_k}) \neq \emptyset$ (on terms over the original signature). We can similarly associate a sequence of positions with any depth 1 position $i$ in any term $s_j$ (by considering the infinite derivation $s_j \to s_{j+1} \to \cdots$).

We now define the *use* of a depth 1 position $i$ in $s_1$.

$$
\begin{aligned}
use(s_1, i) &= \{c\} && \text{if } i_1 \ldots i_k \text{ is the sequence associated with } i,\ s_k|_{i_k} = \{c\}, \\
&&& \text{and } i_k \text{ is an original constant position in } s_k \to s_{k+1} \\
&= \textstyle\bigcup_{j \in J} s_k|_j && \text{if } i_1 \ldots i_k \text{ is the sequence associated with } i, \text{ and } J \text{ is the} \\
&&& \text{set of all positions related to } i_k \text{ in the rule } s_k \to s_{k+1}. \\
&= \textstyle\bigcup_{j \geq 1} s_j|_{i_j} && \text{if the sequence } i_1 \ldots \text{ associated with } i \text{ is infinite}
\end{aligned}
$$

From the definition of *use*, it is easy to see that $use(s_1, i)$ is $R$-joinable and $Reach(use(s_1, i)) \subseteq Reach(s_k|_{i_k})$ for all $k$. An important property of the *use* function is that if $i_1, i_2, \ldots, i_k, \ldots$ is the sequence associated with position $i = i_1$ in $s_1$, then $use(s_k, i_k) = use(s_1, i = i_1)$ for all such $k$.

We wish to map terms $s_i$ over the extended signature to terms $t_i$ over the original signature, and hence we need to find a concrete representation term for each $s_k|_{i_k}$. Therefore, define $Choice(\{c\}) = c$ if $c \in \Sigma_0$, and $Choice(S) = t$ if $S \neq \{c\}$ for any $c \in \Sigma_0$ and $t$ is *any* (selected) term in $Reach(S)$.We map every term $s_i$ of the original infinite derivation into a new term

$$s'_i = f(Choice(use(s_i, 1)), \ldots, Choice(use(s_i, m)))$$

*over the original signature.* Our intention is to show that we have an infinite derivation $s'_1 \to^*_R s'_2 \to^*_R s'_3 \ldots$, proving then that $R$ is nonterminating.

If a rewrite step $s_i \to s_{i+1}$ is done with a rule of $UJoin(R)$, then $s'_i = s'_{i+1}$ by the definition of *use*, and hence $s'_i \to^*_R s'_{i+1}$ trivially. For finishing the proof it will be enough to show that if a rewrite step $s_i \rightarrow s_{i+1}$ is done with a rule of $UPerm$, then $s'_i \rightarrow_R^+ s'_{i+1}$ (note that there are infinitely many steps of this kind in the derivation $s_1 \rightarrow s_2 \rightarrow \ldots$ since by assumption $UJoin(R)$ is terminating). The rule used in this step is precisely $s_i \rightarrow s_{i+1}$ since $UPerm$ is ground. Let $l \rightarrow r$ be the rule in $R$ from which it is constructed. For every variable position $j$ in $l$, let $j_1, \ldots, j_k$ be the positions related to $j$ in the rule $l \rightarrow r$. By the definition of $use$, the sets $use(s_i, j_1), \ldots, use(s_i, j_k)$ are identical, and hence, the terms $Choice(use(s_i, j_1)), \ldots, Choice(use(s_i, j_k))$ are identical. Moreover, if the variable appears in $r$ at position $p$, then all $j_1 \ldots j_k$ go to $p$ in the rewrite rule, and hence $Choice(use(s_{i+1}, p))$ is also the same term. Therefore, rewriting $s'_i$ with $l \rightarrow r$ produces a term, say $s'$, that coincides with $s'_{i+1}$ in all the depth 1 positions that are variable positions in $r$. For the rest of positions, $s'$ contains constants that coincide with the corresponding singleton sets at the same positions in $s_{i+1}$. That is, for any other position $p$, $s'|_p = c$ for some $c \in \Sigma_0$. In this case, $s_{i+1}|_p = \{c\}$. But, it is the case that $c \rightarrow_R^* Choice(use(s_{i+1}, p))$, and hence, $s' \rightarrow_R^* s'_{i+1}$, which proves that $s'_i \rightarrow_R^+ s'_{i+1}$. □

*Example 3.* Consider the rewrite system $R$ and the following infinite $(UPerm(R) \cup UJoin(R))$-derivation from Example 2:

$$f\{a,c\}\{b\}\{b\} \rightarrow^{nr} f\{a,c\}\{b\}\{d\} \rightarrow^{r_2} f\{c\}\{a,c\}\{a\} \rightarrow^{r_1} f\{a,c\}\{a\}\{b\}$$
$$\rightarrow^{r_1} f\{a,c\}\{b\}\{b\} \rightarrow^{nr} \cdots$$

The sequence associated with term $f\{a,c\}\{b\}\{b\}$, call it $s_1$, and position 1 is the infinite sequence $1, 1, 2, 1, 1, 1, 2, 1, 1, \ldots$; whereas the sequence associated with $s_1$ and position 2 is the finite sequence $2, 2$ and the sequence associated with $s_1$ and position 3 is the finite sequence $3, 3$. Therefore, $use(s_1, 1) = \{a, c\}$, $use(s_1, 2) = \{b\}$, and $use(s_1, 3) = \{d\}$. We can set the $Choice$ function so that $Choice(\{a,c\}) = c$, $Choice(\{b\}) = b$, and $Choice(\{d\}) = d$. Lifting the terms using the $Choice(use(\_,\_))$ function, we get the following infinite $R$-derivation:

$$fcbd \rightarrow^{r_2} fcca \rightarrow^{*,nr} fccc \rightarrow^{r_1} fccb \rightarrow^{r_1} fcbb$$
$$\rightarrow^{*,nr} fcbd \rightarrow^{r_2} \cdots$$

Note that we have to apply $\rightarrow^{*,nr}$ steps (here $a \rightarrow^* c$ steps since we chose $c$ as $Choice(\{a,c\})$) to go from an intermediate term (for example, $fcca$) to the lifting of the next term ($fccc$).

### 4.3 Deciding Termination

The following lemma characterizes termination of a rewrite system $R$ that contains only right-constant, right-variable, or flat and right-linear rules.

**Lemma 6.** *$R$ is terminating iff the following three conditions are satisfied:*

1. *There is no insertion rule $x \rightarrow r \in R$.*
2. *The rewrite system $UPerm(R) \cup UJoin(R)$ terminates starting from any flat term of the form $fS_1 \ldots S_n$ where every $S_i$ contains only $R$-joinable constants from $\Sigma$.*

*3. It is not the case that $c \to_R^+ C[c]$ for any constant $c$ and context $C[\_]$.*

*Proof.* $\Rightarrow$: Suppose $R$ is terminating. If either of conditions (1) or (3) are violated, then the rewrite system $R$ is clearly nonterminating. Now suppose conditions (1) and (3) are satisfied but condition (2) is violated and there is an infinite rewriting derivation $s_1 \to s_2 \to s_3 \to \cdots$ with $UPerm \cup UJoin(R)$ starting from a term of the form $s_1 = fS_1 \ldots S_m$, where every $S_i$ is joinable and $S_i \subseteq \Sigma$. Condition (3) implies that $UJoin(R)$ is terminating. This fact together with Lemma 5 implies that $R$ is nonterminating, a contradiction.

$\Leftarrow$: We prove by contradiction. Suppose the three conditions are satisfied but $R$ is nonterminating. We compare nonterminating derivations by the size of their initial terms. For the case of two derivations starting from constants, we compare them by comparing the constants with the following ordering: $d$ is smaller than $c$ if $c \to_R^+ C[d]$ for some context $C[\_]$ (by condition (3) this is a well founded ordering). We consider a minimal nonterminating derivation:

$$s = s_1 \to s_2 \to s_3 \to \cdots$$

Consider the top rewrite steps in this derivation. If any of these top steps are collapsing, then they can be commuted with the next rewrite step and moved to the right. Repeatedly doing this would result in an infinite derivation without top collapsing steps and with the same initial term. We can, therefore, assume that there are no top collapsing steps in the above derivation. Moreover, there are no applications of rewrite rules of the form $l \to c$ at the top, since otherwise, from that point on we obtain a smaller derivation (either by the size or the constant ordering). This observation, along with condition (1), means that we can assume that all top steps in the above infinite derivation have to be applications of the permutation rules $f \ldots \to f \ldots$, with the exception that the first top step can be the application of a rule of the form $c \to f \ldots$.

There are two cases:
(a) there are finitely many top rewrite steps, or
(b) there are infinitely many top rewrite steps.

Case (a): Suppose there are no top rewrite steps applied after reaching term $s_i$. Clearly, there is an infinite derivation starting from some subterm $s_i|_p$ of $s_i$ where $p \in \{1, \ldots, m\}$. Since all root rewrites are done using left-constant or permutation rules, it follows that the term $s_i|_p$ is reachable from some subterm $s_1|_{p'}$ with $p' \in \{1, \ldots, m\}$, or it is reachable from some constant $c$ such that $s_1 \to_R^+ C[c]$. In the former case, there is an infinite derivation starting from a strictly smaller term $s_1|_{p'}$. In the latter case, there is a smaller infinite derivation starting from $c$ (either in the size or the constant ordering).

Case (b): In this case we assume that $s_1$ is not a constant. Otherwise, we consider the same derivation but starting from $s_2$. Lemma 4 shows that there is an infinite $(UPerm(R) \cup UJoin(R))$-derivation starting from a ground term $fS_1 \ldots S_m$, where $S_i \subseteq \Sigma_0$ are $R$-joinable. This contradicts Condition (2). □

*Example 4.* The rewrite system $R = \{fxxx \to fxc_1c_2, c_2 \to c_1\}$ is nonterminating because the rewrite system $\{f\{c_1\}\{c_1\}\{c_2\} \to f\{c_1, c_2\}\{c_1\}\{c_2\}\ ,\ \{c_1, c_2\} \to$

$\{c_1\}\}$, which is contained in the union of $UJoin(R)$ and $UPerm(R)$, does not terminate starting from $f\{c_1\}\{c_1\}\{c_2\}$.

Finally, we show that the three conditions characterizing termination of $R$ can be decided using the decidability of $R$-reachability and $R$-joinability and the decidability of termination of ground TRSs. This result subsumes our previous termination decidability result [9].

**Theorem 2.** *The termination property for TRS's containing only shallow right-linear rules, arbitrary collapse rules, and right-ground rules is decidable.*

*Proof.* Using Theorem 1, any such TRS can be transformed to a TRS $R$ that contains only flat right-linear rules, arbitrary collapse rules, and right-constant rules, while preserving termination. Hence, decidability reduces to checking the three conditions of Lemma 6.

Condition (1) is trivially checkable. For the decidability of condition (3), we consider any constant $c$ and distinguish two cases: checking if $c \to_R^+ c$ and checking if $c \to_R^* C[c]$ for some non-empty context $C[\_]$. For the first case, note that, since $Vars(r) \subseteq Vars(l)$ is satisfied, the number of different terms reachable from $c$ in one rewrite step is finite; and hence, we can check if $c$ is reachable from every one of them by the decidability of $R$-reachability. For the second case, note that, since $R$ is regularity-preserving, the set of terms reachable from $c$ is recognizable. We can now check condition (3) by checking emptiness of the intersection of this set with the set of terms in which $c$ occurs at non-root position, which is recognizable, too.

For the decidability of condition (2), note that the rewrite systems $UJoin(R)$ and $UPerm(R)$ restricted to set-constants $S$ s.t. $S \subseteq \Sigma_0$ can be constructed, due to the fact that $R$-reachability and $R$-joinability are decidable, and that the number of different $S$ s.t. $S \subseteq \Sigma_0$ is finite. Now, this case reduces to checking termination of a ground TRS, which is decidable. □

## 5 Conclusion

In this paper, we showed that termination is decidable for rewrite systems that contain right-ground, collapsing, or shallow right-linear rewrite rules. The proof is especially elegant since it is modular over the decidability results for reachability and joinability [17]. We also prove some properties about rewriting using shallow right-linear TRSs, which are used to prove the main results of this paper. Using these intuitions, we have shown elsewhere [10] that confluence is decidable for shallow and right-linear rewrite systems.

It will be interesting to explore the possibility of extending the class of rewrite systems without compromising decidability of termination. Another direction for future work would be investigating the termination of rewriting modulo certain axioms such as associativity and commutativity.

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