Decreasing Diagrams: Two Labels Suffice

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Overview

1. Preliminaries
   - Abstract reduction systems and confluence (CR)
   - Decreasing Church-Rosser (DCR) and decreasing diagrams
   - Cofinality property (CP)
   - Dependencies between CR, DCR and CP
2. Two labels suffice
   - Departing question: DCR hierarchy
   - Proof sketch for $CP \Rightarrow DCR_2$
   - Dependencies between properties (updated)
3. Further results: commutation

Abstract reduction systems and confluence (CR)

- *ARS* $\mathcal{A} = (A, \to)$ with $\to \subseteq A \times A$
- $\mathcal{A}$ is *countable* if A is
- $(A, \to)$ is *confluent* (CR) if

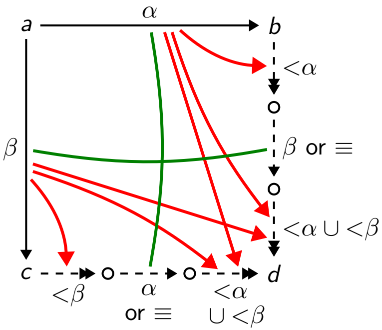
$$(a \twoheadrightarrow b \wedge a \twoheadrightarrow c) \Rightarrow \exists d \in A \, (b \twoheadrightarrow d \wedge c \twoheadrightarrow d)$$

- *indexed ARS* $\mathcal{A} = (A, \{\to_\alpha\}_{\alpha \in I})$ letting $\to = \bigcup_{\alpha \in I} \to_\alpha$

Decreasing Church-Rosser (*DCR*) and decreasing diagrams

Definition 1 (Decreasing Church–Rosser [4])

$\mathcal{A} = (A, \to)$ is *decreasing Church–Rosser (DCR)* if it equals $\mathcal{B} = (A, \{\to_\alpha\}_{\alpha \in I})$ indexed by a well-founded partial order $(I, <)$ such that every peak $c \leftarrow_\beta a \to_\alpha b$ can be joined *decreasingly*.



Theorem 2 (Decreasing Diagrams – De Bruijn [1] & Van Oostrom [4])

$$DCR \Rightarrow CR$$

Cofinality property (CP)

Definition 3 (Cofinal Reduction)

Let $\mathcal{A} = (A, \rightarrow)$ be an ARS. A finite or infinite reduction sequence $b_0 \rightarrow b_1 \rightarrow b_2 \rightarrow \cdots$ is *cofinal in* $\mathcal{A}$ if $a \in A$ implies $a \twoheadrightarrow b_i$ for some i.

Definition 4 (Cofinality Property)

An ARS $\mathcal{A} = (A, \rightarrow)$ has the *cofinality property* (*CP*) if for every $a \in A$, there exists a reduction sequence $a \equiv b_0 \rightarrow b_1 \rightarrow b_2 \rightarrow \cdots$ that is cofinal in $\mathcal{A}|_{\{b \mid a \twoheadrightarrow b\}}$.

$$\begin{tikzcd}
a \arrow[r, dashed] \arrow[drr] & b_1 \arrow[r, dashed] & \cdots \arrow[r, dashed] & b_i \arrow[r, dashed] & \cdots \\
 & & \circ \arrow[ur, dashed] & &
\end{tikzcd}$$

Dependencies between CR, DCR and CP

For countable ARSs, the relevant properties coincide:

$$
\begin{tikzcd}
CP \arrow[r] & DCR \arrow[r] & CR \arrow[ll, bend right]
\end{tikzcd}
$$

For uncountable systems, the situation is as follows:

$$
\begin{tikzcd}
CP \arrow[r] & DCR \arrow[r] \arrow[l, bend left, "\textcolor{red}{X}"] & CR \arrow[ll, bend right, "\textcolor{red}{X}"'] \arrow[l, bend left, "\textcolor{blue}{?}"]
\end{tikzcd}
$$

Example 5 (Counter-example to $DCR \Rightarrow CP$)

Let A be the set of finite subsets X of the line $\mathbb{R}$. Consider the reduction rule $X \to X \cup \{x\}$ for $x \notin X$. The uncountable ARS $(A, \to)$ is DCR, but not CP.

Whether $CR \Rightarrow DCR$ for uncountable systems is a long-standing open problem in abstract rewriting.

Departing question: *DCR* hierarchy

Definition 6 (DCR_α)

For ordinals α, let DCR_α denote the class of ARSs that can be shown to satisfy DCR using the label set $\{\beta \mid \beta < \alpha\}$. (With $<$ the usual order on ordinals.)

- Do we have strict inclusions $DCR_\alpha \subset DCR_\beta$ for all $\alpha < \beta$?
- If not, what does the hierarchy look like?

Our main result:

Theorem 7 (Two Labels Suffice – Klop, Endrullis & Overbeek [2])

$$CP \Rightarrow DCR_2$$

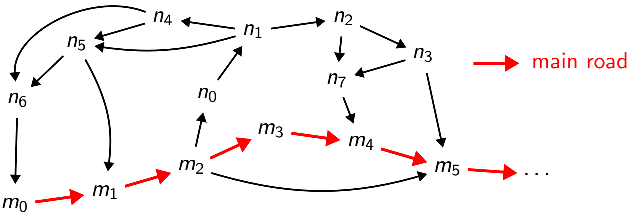
- Thus one easily obtains $DCR = DCR_2$ for the countable case.
- Our proof is an adaptation of Van Oostrom's proof for $CP \Rightarrow DCR$ [3, Proposition 14.2.30, p. 766].

$CP \Rightarrow DCR_2$: proof sketch (1/4)

Lemma 8 ($CP \Rightarrow CP^{\leftrightarrow^*}$ – Mano, 1993)

Let $\mathcal{A} = (A, \to)$ be a confluent ARS and $a \in A$. If a rewrite sequence is cofinal in $\mathcal{A}|_{\{b \mid a \twoheadrightarrow b\}}$, then it is also cofinal in $\mathcal{A}|_{\{b \mid a \leftrightarrow^* b\}}$.

Thus CP implies that there exists a main road in every weakly connected component (w.r.t. $\leftrightarrow^*$).



We focus on a single component ARS $\mathcal{A} = (A, \to)$ satisfying CP, and let M denote a fixed acyclic main road in $\mathcal{A}$.

$CP \Rightarrow DCR_2$: proof sketch (2/4)

Our labelling function presupposes the following notions.

- $d(a)$ is the distance of a to the main road M
- $>$ is a linear order on A

Definition 9 (Minimizing Step)

A step $a \to b$ is *minimizing* if

(i) $d(a) = d(b) + 1$ and

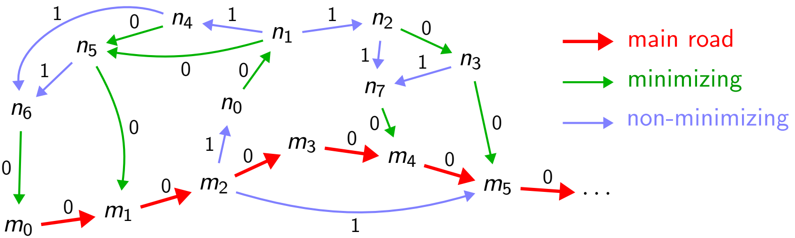
(ii) $b' \geq b$ for every step $a \to b'$ with $d(b') = d(b)$.

Remark: $>$ exists by the Well-Ordering Theorem, or by construction for countable systems

$CP \Rightarrow DCR_2$: proof sketch (3/4)

We now label steps $a \to b$ with 0 or 1 as follows:

$$a \to_0 b \iff a \to b \text{ is on } M \text{ or minimizing}$$
$$a \to_1 b \iff a \to b \text{ is not on } M \text{ and not minimizing}$$



$CP \Rightarrow DCR_2$: proof sketch (4/4)

We show DCR. There are three cases for the peaks:

$$\begin{tikzcd} a \arrow[r, "0"] \arrow[d, "0"'] & b \arrow[d, dashed, "\equiv"] \\ c \arrow[r, dashed, "\equiv"'] & d \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "1"] \arrow[d, "0"'] & b \arrow[d, dashed, two heads, "0"] \\ c \arrow[r, dashed, two heads, "0"'] & d \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "1"] \arrow[d, "1"'] & b \arrow[d, dashed, two heads, "0"] \\ c \arrow[r, dashed, two heads, "0"'] & d \end{tikzcd}$$

- $\rightarrow_0$ is deterministic
- there exist 0-labelled paths from any point to M
- any two points on M can be joined by 0-reductions $\square$

Dependencies between CR, DCR, CP and DCR_2

For countable ARSs, we now have:

$$
\begin{tikzcd}
CP \arrow[r] & DCR_2 \arrow[r] & DCR \arrow[r] & CR \arrow[lll, bend right]
\end{tikzcd}
$$

And for uncountable systems:

$$
\begin{tikzcd}
CP \arrow[r] & DCR_2 \arrow[r] \arrow[l, bend right, "\times"'] & DCR \arrow[r] \arrow[l, bend right, "?"'] & CR \arrow[l, bend right, "?"'] \arrow[ll, bend right=60, "?"']
\end{tikzcd}
$$

The implications $DCR \Rightarrow DCR_2$ and $CR \Rightarrow DCR_2$ are new open problems for the uncountable case.

Further results: commutation (1/2)

Relation $\to$ *commutes with* $\rightsquigarrow$ in an ARS $(A, \to, \rightsquigarrow)$ if:

$$
\begin{tikzcd}
& a \arrow[dl, rightsquigarrow] \arrow[dr] & \\
b \arrow[dr, dashed] & & c \arrow[dl, rightsquigarrow, dashed] \\
& d &
\end{tikzcd}
$$

DCR can be used to prove commutation (although it is *incomplete*):

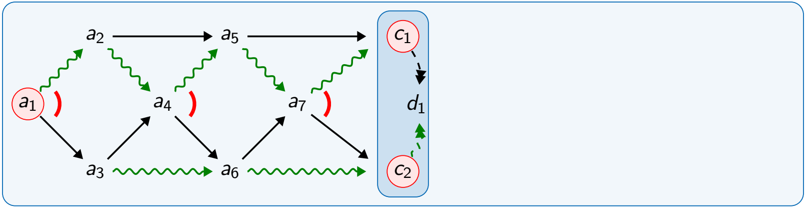
$$
\begin{tikzcd}
a \arrow[rrr, rightsquigarrow, "\alpha"] \arrow[ddd, "\beta"'] & & & b \arrow[d, dashed, twoheadrightarrow, "<\alpha"] \\
& & & \circ \arrow[d, dashed, "\beta \text{ or } \equiv"] \\
& & & \circ \arrow[d, dashed, twoheadrightarrow, "<\alpha \cup <\beta"] \\
c \arrow[r, rightsquigarrow, dashed, twoheadrightarrow, "<\beta"'] & \circ \arrow[r, rightsquigarrow, dashed, "\alpha \text{ or } \equiv"'] & \circ \arrow[r, rightsquigarrow, dashed, twoheadrightarrow, "<\alpha \cup <\beta"'] & d
\end{tikzcd}
$$

Question: do we have $DCR = DCR_2$ for commutation?

Further results: commutation (2/2)

Theorem 10 (Lower *DCR* hierarchy for commutation)

For commutation, $DCR_\alpha \subset DCR_\beta$ for all ordinals $\alpha < \beta \leq \omega$



- **plan:** extend system to require additional label
- assume $c_2 \rightsquigarrow^* d_1 \leftarrow^* c_1$ with two steps $\geq n$ on one of the reductions
- peaks from a_1, a_4 and a_7 each contain a step $\geq n+1$
- hence $a_1 \rightsquigarrow^* c_1$ and $a_1 \rightarrow^* c_2$ with three steps $\geq n+1$

Further results: commutation (2/2)

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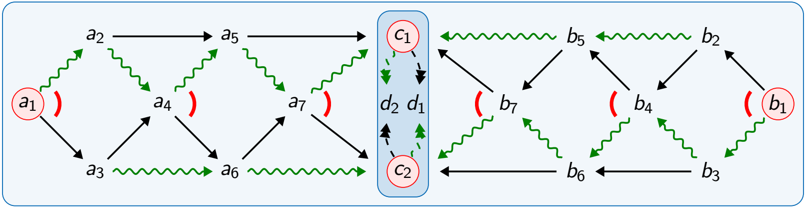

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- hence $a_1 \rightsquigarrow^* c_1$ and $a_1 \rightarrow^* c_2$ with three steps $\geq n+1$
- **or:** symmetric for $c_1 \rightsquigarrow^* d_2 \leftarrow^* c_2$, b_1 to b_7 $\square$

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