Exercise 1 : Programming in Curien-Herbelin-Wadler's calculus

1. What is the general shape of typed commands in normal form?
2. From now on we assume that all the reductions are CBN, and we also forbid the continuation construct corresponding to "$\mathsf{let}\ldots\mathsf{in}\ldots$", i.e. $\mu x.c$.
   Consider two terms t and $\mu\alpha.\langle t \bullet \alpha\rangle$ with $\alpha \notin \mathsf{FV}(t)$. Show that for all continuations e, $\langle t \bullet e\rangle$ and $\langle \mu\alpha.\langle t \bullet \alpha\rangle \bullet e\rangle$ bi-simulate each other for the relation $\longrightarrow^*_{\mathsf{CBN}}$.
3. We now add to CBN-reductions the CBN η-rule: $\mu\alpha.\langle t \bullet \alpha\rangle \longrightarrow_{\eta\mathsf{CBN}} t$ if $\alpha \notin \mathsf{FV}(t)$.
   Let a be a simple atomic type, $\Gamma = (x{:}a, f{:}a{\to}a)$, and $\Delta_n = (\alpha_1{:}a, \ldots, \alpha_n{:}a)$.
   (a) Describe all continuations e in normal form such that $\Gamma\ ;\ e{:}a \vdash \Delta_n$
   (b) Describe all commands c in normal form such that $c{:}(\Gamma \vdash \Delta_n)$
   and all terms t in normal form such that $\Gamma \vdash t{:}a\ ;\ \Delta_n$
4. Describe all terms t in normal form such that $\Gamma \vdash t{:}a\ ;$ and every sub-term $\mu\alpha.c$ of t is such that α has exactly one free occurrence in c.
5. Give an encoding $\underline{n}$ of natural numbers n as terms $\lambda x.\lambda f.t$, where t is of the form described above.
6. Using that encoding,
   (a) Write a term suc such that $\langle \mathsf{suc} \bullet \underline{n}{::}\, e\rangle \longrightarrow^* \langle \underline{n+1} \bullet e\rangle$.
   (b) Write a term plus such that $\langle \mathsf{plus} \bullet \underline{n}{::}\, \underline{q}{::}\, e\rangle \longrightarrow^* \langle \underline{n+q} \bullet e\rangle$.
   (c) Write a term times such that $\langle \mathsf{times} \bullet \underline{n}{::}\, \underline{q}{::}\, e\rangle \longrightarrow^* \langle \underline{n \times q} \bullet e\rangle$.
7. Write three terms $\mathsf{true}, \mathsf{false}, \mathsf{if}$ such that
   $\langle \mathsf{if} \bullet \mathsf{true}{::}\, t_1{::}\, t_2{::}\, e\rangle \longrightarrow^*_\beta \langle t_1 \bullet e\rangle$ and $\langle \mathsf{if} \bullet \mathsf{false}{::}\, t_1{::}\, t_2{::}\, e\rangle \longrightarrow^*_\beta \langle t_2 \bullet e\rangle$
8. Give an (untyped) command that is not strongly normalising.

Exercise 2 : Translating λ into Curien-Herbelin-Wadler's calculus

$$\begin{array}{lcl} \overline{x}^{\mathsf{V}} & := & x \\ \overline{\lambda x.M}^{\mathsf{V}} & := & \lambda x.\overline{M} \end{array} \qquad \begin{array}{l} \overline{V\ M_1\ \ldots\ M_n} := \mu\alpha.\langle \overline{V}^{\mathsf{V}} \bullet \overline{M_1}{::}\ldots\overline{M_n}{::}\,\alpha\rangle \\ \quad \text{where } V \text{ is not an application and } n \geq 0 \end{array}$$

1. Prove that $\mu\alpha.\langle \overline{M} \bullet \overline{M_1}{::}\ldots\overline{M_n}{::}\,\alpha\rangle \longrightarrow^* \overline{M\ M_1\ \ldots\ M_n}$ for $n \geq 1$.
2. Prove that $\left\{\overline{N}/_x\right\}\overline{M} \longrightarrow^* \overline{\{N/_x\}M}$.
3. Prove that if $M \longrightarrow_\beta N$ then $\overline{M} \longrightarrow^* \overline{N}$.