

CS3202: Logic, Specification and Verification

CS3202-LSV 2006-07

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Lecture 7 (12/03/2007): Induction

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 We know \mathbb{B} has two elements.
 $\mathbb{C}\mathsf{omp} = \{\texttt{LT}, \texttt{EQ}, \texttt{GT}\}$ $\mathbb{C}\mathsf{omp}$ has three (and some other stuff).

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But we know this about \mathbb{N} (roughly speaking, Peano's 1887 axioms 1–5):

1. $0 \in \mathbb{N}$ "there is a natural number"2. if $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$ "there is a successor (function)"3. if n + 1 = m + 1 then n = m"the successor function is injective"4. $0 \neq n + 1$ for any n(discrimination)otherwise put: if 0 = 1 then anything might be true

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 $\begin{array}{ccc} & [m:\mathbb{N}] & [P\ m] \\ & & \vdots \\ P\ 0 & P\ (m+1) \\ \hline \forall n:\mathbb{N},P\ n \end{array}$

$$[m:\mathbb{N}]$$
 $[Pm]$ Rule of mathematical induction on \mathbb{N} : $P 0$ $P(m+1)$ $\forall n:\mathbb{N}, Pn$ $\forall n:\mathbb{N}, Pn$

Alteri $[m \ n : \mathbb{N}] \qquad [P \ m] \qquad [n = m + 1]$ [ד א Г

$$\begin{bmatrix} n:\mathbb{N} \end{bmatrix} \quad \begin{bmatrix} n=0 \end{bmatrix} \qquad \begin{bmatrix} m,n:\mathbb{N} \end{bmatrix} \quad \begin{bmatrix} P m \end{bmatrix} \quad \begin{bmatrix} n=m+1 \\ \vdots \\ P n \qquad \qquad P n \end{bmatrix}$$

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$$\begin{array}{ccc} [m:\mathbb{N}] & [P\ m] \\ \vdots \\ \\ \text{Rule of mathematical induction on } \mathbb{N}: & \underbrace{P\ 0 & P\ (m+1)}_{\forall n:\ \mathbb{N},\ P\ n} \\ \\ \text{Alternative with explicit equations} \\ [n:\mathbb{N}] & [n=0] & [m,n:\mathbb{N}] & [P\ m] & [n=m+1] \\ \vdots & & & \vdots \\ P\ n & & & P\ n \end{array}$$

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Alternative as an implication

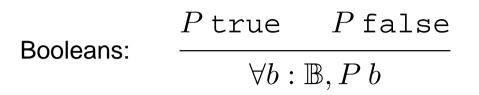
$$P(0) \Rightarrow [\forall m : \mathbb{N}, P(m) \Rightarrow P(m+1)] \Rightarrow \forall n : \mathbb{N}, P(n)$$

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$$P(0) \Rightarrow [\forall m : \mathbb{N}, P(m) \Rightarrow P(m+1)] \Rightarrow \forall n : \mathbb{N}, P(n)$$



In each case, the universal conclusion $\forall d: D, P d$ follows from

- one case for each way of forming a new data item -via constructors
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Questions?

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