CS3202: Logic, Specification and Verification

CS3202-LSV 2006–07

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Lecture 6 (02/03/2007):

Predicate Logic —

Semantical notions, equality & conclusion

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Questions?