

CS3202: Logic, Specification and Verification

CS3202-LSV 2006-07

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Lecture 5 (26-27/02/2007):

Predicate Logic

JHM+SL: CS3202 Lecture 5 Slide 0

- Socrates is a man
- All men are mortal
- Socrates is mortal

- Socrates is a man: H(S)
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or *quantifiers+variable*, $\forall x$ (arity 1), $\exists x$ (arity 1)

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x is **bound** by the quantifier.

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- looks (usually) just what you think
- but beware!
- need to define what it is to be *free* and *bound* in an expression, formula, etc.
- $FV(\forall x : A.M) = FV(M) \setminus \{x\}$
- lots of room to make mistakes...so go to the machine for assistance...
- Note: a variable can be substituted for a term which may refer to other variables (possibly *be* a variable). Example:
 ∀m, ¬(n = 0){n → m + 1} becomes ∀m, ¬(m + 1 = 0)
 In other words n is not (yet) substituted for a "value", i.e. a term without (free) variables (all leaves are constants).

• on terms

$$x\{x \mapsto t\} = t$$

$$y\{x \mapsto t\} = y$$

$$f(u_1, \dots, u_n)\{x \mapsto t\} = f(u_1\{x \mapsto t\}, \dots, u_n\{x \mapsto t\})$$

on wff

• • •

$$(p(u_1, \dots, u_n))\{x \mapsto t\} = p(u_1\{x \mapsto t\}, \dots, u_n\{x \mapsto t\})$$
$$(A \land B)\{x \mapsto t\} = (A\{x \mapsto t\}) \land (B\{x \mapsto t\})$$

$$\begin{array}{ll} (\forall y, A)\{x \mapsto t\} & = \forall y, (A\{x \mapsto t\}) & x \neq y, y \notin \textit{FV}(t) \\ (\exists y, A)\{x \mapsto t\} & = \exists y, (A\{x \mapsto t\}) & x \neq y, y \notin \textit{FV}(t) \end{array}$$

• We extend Natural Deduction with the following rules:

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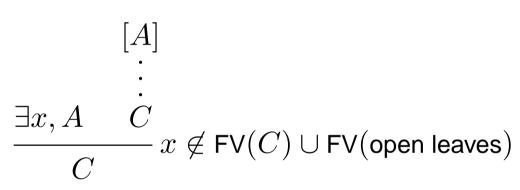
● ∃-introduction:

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• \exists -introduction:

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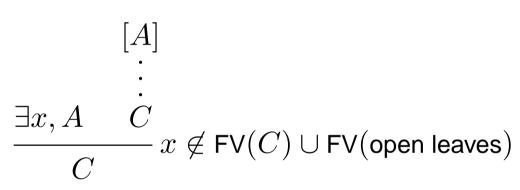
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• \exists -introduction:

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Intuition about \exists -elim: an alternative?

• Standard \exists -elimination:

$$\begin{array}{c} [A] \\ \vdots \\ \exists x, A \quad C \\ \hline C \end{array} x \not\in \mathrm{FV}(C) \cup \mathrm{FV}(\text{open leaves}) \end{array}$$

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• Writing P(x) for a wff A depending on x,

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• Worse:

$$\frac{\exists x, P(x) \quad \frac{[P(x)]}{P(x)}}{\frac{P(x)}{\forall x, P(x)}}$$

is a proof for $\exists x, P(x) \vdash \forall x, P(x)$

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 $\bullet \ \forall x, (p(x) \Rightarrow q(x)) \vdash (\forall y, p(y)) \Rightarrow \forall z, q(z)$

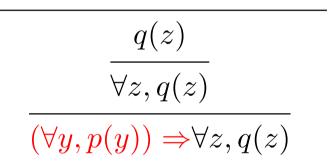
• $\forall x, (p(x) \Rightarrow q(x)) \vdash (\forall y, p(y)) \Rightarrow \forall z, q(z)$ Proof:

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• $\forall x, (p(x) \Rightarrow q(x)) \vdash (\forall y, p(y)) \Rightarrow \forall z, q(z)$ Proof:

$$\frac{\overline{\forall z, q(z)}}{(\forall y, p(y)) \Rightarrow \forall z, q(z)}$$

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$p(z) \Rightarrow q(z)$	p(z)
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$\overline{orall z,q(z)}$	
$(\forall y, p(y)) \Rightarrow \forall z, q$	q(z)

	$q(x)) \vdash (\forall y, p(y)) \Rightarrow \forall z$,q(z)
Proof:	$\forall x, (p(x) \Rightarrow q(x))$	
	$p(z) \Rightarrow q(z)$	p(z)
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since $p(z) \Rightarrow q(z)$ is $(p(x) \Rightarrow q(x)) \{ x \mapsto z \}$

• $\forall x, (p(x) \Rightarrow q(x)) \vdash (\forall y, p(y)) \Rightarrow \forall z, q(z)$ Proof:

$\forall x, (p(x) \Rightarrow q(x))$	$[\forall y, p(y)]$
$p(z) \Rightarrow q(z)$	p(z)
$\overline{q(z)}$	
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$\overline{(\forall y, p(y)) \Rightarrow} \forall z$	$\overline{q}, q(z)$

since $p(z) \Rightarrow q(z)$ is $(p(x) \Rightarrow q(x))\{x \mapsto z\}$ and p(z) is $(p(y))\{y \mapsto z\}$

• $\forall x, (p(x) \Rightarrow q(x)) \vdash (\forall y, p(y)) \Rightarrow \forall z, q(z)$ Proof:

$\forall x, (p(x) \Rightarrow q(x))$	$[\forall y, p(y)]$
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Note that the step \star is correct because

 $z \not\in \mathrm{FV}(\forall x, (p(x) \Rightarrow q(x))) \cup \mathrm{FV}(\forall y, p(y))$

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• $\exists x, (p(x) \lor q(x)) \vdash (\exists y, p(y)) \lor (\exists z, q(z))$

Proof:

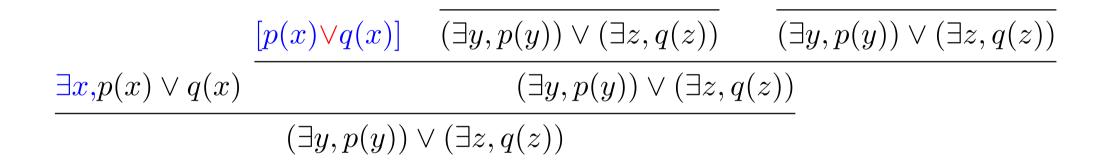
 $(\exists y, p(y)) \lor (\exists z, q(z))$

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Proof:

$$\frac{\exists x, p(x) \lor q(x)}{(\exists y, p(y)) \lor (\exists z, q(z))}$$

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• $\exists x, (p(x) \lor q(x)) \vdash (\exists y, p(y)) \lor (\exists z, q(z))$

Proof:

since p(x) is $(p(y))\{y\mapsto x\}$

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Proof:

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$$(\exists y, p(y))\lor (\exists z, q(z))$$

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$$\frac{\exists x, p(x)\lor q(x)}{(\exists y, p(y))\lor (\exists z, q(z))} \times (\exists y, p(y))\lor (\exists z, q(z))}{(\exists y, p(y))\lor (\exists z, q(z))} \times$$

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 $x \not\in \mathrm{FV}((\exists y, p(y)) \lor (\exists z, q(z)))$ and at that point there is no open assumption

• $\exists x, (p(x) \lor q(x)) \vdash (\exists y, p(y)) \lor (\exists z, q(z))$

Proof:

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$$\frac{\exists x, p(x)\lor q(x)}{(\exists y, p(y))\lor (\exists z, q(z))} \times (\exists y, p(y))\lor (\exists z, q(z))}{(\exists y, p(y))\lor (\exists z, q(z))} \times$$

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• $(\exists y, p(y)) \lor (\exists z, q(z)) \vdash \exists x, (p(x) \lor q(x))$

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• $(\exists y, p(y)) \lor (\exists z, q(z)) \vdash \exists x, (p(x) \lor q(x))$ Proof:

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• $(\exists y, p(y)) \lor (\exists z, q(z)) \vdash \exists x, (p(x) \lor q(x))$ Proof:

$$\frac{[\exists y, p(y)] \ \exists x, (p(x) \lor q(x))}{\exists x, (p(x) \lor q(x))} \qquad \exists x, (p(x) \lor q(x)) \\ \exists x, (p(x) \lor q(x)) \qquad \exists x, (p(x) \lor q(x)) \end{cases}$$

• $(\exists y, p(y)) \lor (\exists z, q(z)) \vdash \exists x, (p(x) \lor q(x))$ Proof:

 $\frac{[p(y)]}{p(y) \lor q(y)}$ $(\exists y, p(y)) \lor (\exists z, q(z)) \qquad \boxed{\exists y, p(y)]} \ \exists x, (p(x) \lor q(x)) \qquad \qquad \boxed{\exists x, (p(x) \lor q(x))} \qquad \qquad \boxed{$

since $p(y) \lor q(y)$ is $(p(x) \lor q(x)) \{ x \mapsto y \}$

• $(\exists y, p(y)) \lor (\exists z, q(z)) \vdash \exists x, (p(x) \lor q(x))$ Proof:

since $p(y) \lor q(y)$ is $(p(x) \lor q(x)) \{ x \mapsto y \}$

• $(\exists y, p(y)) \lor (\exists z, q(z)) \vdash \exists x, (p(x) \lor q(x))$ Proof:

since $p(y) \lor q(y)$ is $(p(x) \lor q(x)) \{ x \mapsto y \}$

• $(\exists y, p(y)) \lor (\exists z, q(z)) \vdash \exists x, (p(x) \lor q(x))$ Proof:

since $p(y) \lor q(y)$ is $(p(x) \lor q(x))\{x \mapsto y\}$ and $p(z) \lor q(z)$ is $(p(x) \lor q(x))\{x \mapsto z\}$ • $(\exists y, p(y)) \lor (\exists z, q(z)) \vdash \exists x, (p(x) \lor q(x))$ Proof:

since $p(y) \lor q(y)$ is $(p(x) \lor q(x)) \{x \mapsto y\}$ and $p(z) \lor q(z)$ is $(p(x) \lor q(x)) \{x \mapsto z\}$ Again, the two \exists -elim are correct: the bound variable y (resp. z) is not free in the conclusion $\exists x, (p(x) \lor q(x))$ & no open assumption • $(\exists y, p(y)) \lor (\exists z, q(z)) \vdash \exists x, (p(x) \lor q(x))$ Proof:

since $p(y) \lor q(y)$ is $(p(x) \lor q(x)) \{x \mapsto y\}$ and $p(z) \lor q(z)$ is $(p(x) \lor q(x)) \{x \mapsto z\}$ Again, the two \exists -elim are correct: the bound variable y (resp. z) is not free in the conclusion $\exists x, (p(x) \lor q(x))$ & no open assumption

Questions?

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