CS3202: Logic, Specification and Verification

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Lecture 2 (12/02/2007):

Propositional logic —

The notion of proof in Natural Deduction

Review of propositional logic

- A *syntax* to express statements (wwf)
- A *semantics* to interpret the statements in the world $\{True, False\}$ given by *valuations*

Review of propositional logic: Consequence

- First approach: via the *semantics*

  use the interpretation in the world to determine whether formulae are true or false

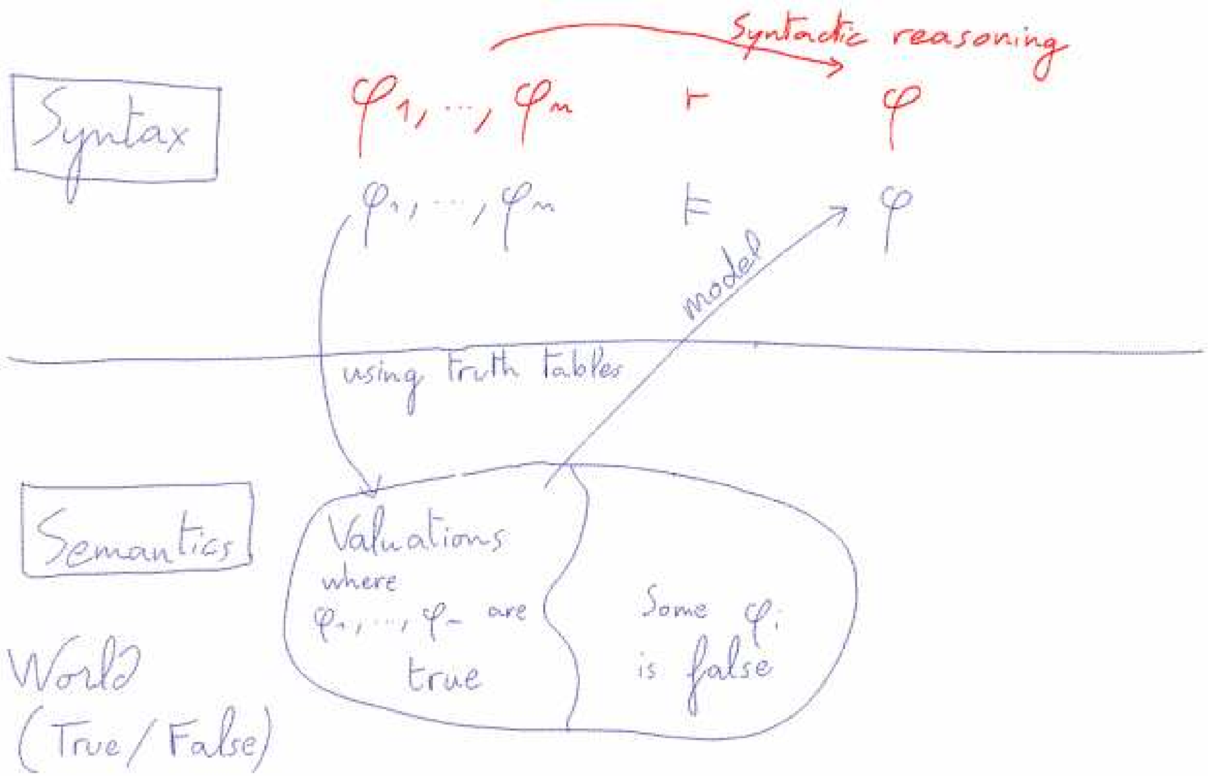
  Tautology: wff with value true in *all* *valuations*

  Even better: use the interpretation in the world to determine whether a formula ϕ is a *logical consequence* from other formulae $\phi_1, \ldots, \phi_n$:

  every model of $\phi_1, \ldots, \phi_n$ is a model of ϕ

$$\phi_1, \ldots, \phi_n \models \phi$$

Review of propositional logic: Consequence



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Review of propositional logic: Consequence

- Second approach: directly in the *syntax* . . . *Why?*

  design a syntactic relation

$$\phi_1, \ldots, \phi_n \vdash \phi$$

  via the notion of *proof* .

  This is defined by a proof-theoretic formalism, e.g. *Natural deduction* (but there are others)

  Question: Is $\phi_1, \ldots, \phi_n \models \phi$ equivalent to $\phi_1, \ldots, \phi_n \vdash \phi$?

Natural deduction: the general idea

- Proof-theoretic formalism in which:

  A proof is a (labelled & well-formed) tree.

  – Nodes are labelled with wff. Example:

$$\dfrac{a \qquad \dfrac{b}{b \vee c}}{a \wedge (b \vee c)}$$

  – Internal nodes and subtrees follow rules: *inference rules*. Example:

$$\dfrac{A \qquad B}{A \wedge B}$$

Schematic rules and instances

- Schema:

$$\frac{A \qquad B}{A \wedge B}$$

  where A, B range over wff

- Instances (examples):

$$\frac{c \qquad c'}{c \wedge c'} \qquad \frac{\neg c \qquad \neg c'}{\neg c \wedge \neg c'} \qquad \cdots$$

  for the particular atoms c and c'

Natural deduction: the syntactic consequence

- Proof-theoretic formalism in which:

  A proof is a (labelled & well-formed) tree.

  We define the relation $\phi_1, \ldots, \phi_n \vdash \phi$ as:

  "there exists a (proof-)tree whose leaves (a.k.a. *hypotheses*) are labelled with wff among $\phi_1, \ldots, \phi_n$ and whose root (a.k.a. *conclusion*) is labelled with ϕ"

Natural deduction: the actual rules for $\wedge$

- $\wedge$-introduction:

$$\frac{A \qquad B}{A \wedge B}$$

$\wedge$-elimination:

$$\frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

Natural deduction: the actual rules for $\Rightarrow$

- $\Rightarrow$-introduction:

$$\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \Rightarrow B}$$

A is *discharged*.

$\Rightarrow$-elimination (a.k.a. *Modus Ponens*):

$$\frac{A \Rightarrow B \qquad A}{B}$$

Natural deduction: the actual rules for $\Rightarrow$

- Need to adapt the notion of proof:

  Labelled & well-formed tree + subset of leaves (active leaves).

  Discharge A = remove from the set some leaves of the subtree labelled with A

- We define the relation $\phi_1, \ldots, \phi_n \vdash \phi$ as:

  "there exists a (proof-)tree whose *active* leaves are labelled with wff among $\phi_1, \ldots, \phi_n$ and whose root is labelled with ϕ"

Natural deduction: the actual rules for $\perp$, $\neg$ and $\vee$

- $\perp$-introduction: none

$\perp$-elimination: $\dfrac{\perp}{A}$

- $\neg$-introduction: $\dfrac{\begin{array}{c}[A] \\ \vdots \\ \perp\end{array}}{\neg A}$

$\neg$-elimination: $\dfrac{A \qquad \neg A}{\perp}$

- $\vee$-introduction: $\dfrac{A}{A \vee B}$ $\dfrac{B}{A \vee B}$

$\vee$-elimination: $\dfrac{A \vee B \qquad \begin{array}{c}[A] \\ \vdots \\ C\end{array} \qquad \begin{array}{c}[B] \\ \vdots \\ C\end{array}}{C}$

Natural deduction: Soundness

- $\phi_1, \ldots, \phi_n \vdash \phi$ implies $\phi_1, \ldots, \phi_n \models \phi$?

  We prove it by induction on the height of tree. The inductive step amounts to analysing whether each inference rule is *correct*.

- Later: a lecture on *induction*.

  (structural) induction as the reasoning counterpart to function definition by (structural) recursion

Natural deduction: Completeness

- $\phi_1, \ldots, \phi_n \models \phi$ implies $\phi_1, \ldots, \phi_n \vdash \phi$?

  Are the rules enough to characterise semantic consequence?

  We shall see tomorrow.

Questions?