Strong Normalisation of λLJQ

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Abstract

The purpose of this paper is to prove the claim in [DL06, DL07] that typed terms of λLJQ are terminating.

Contents

1 Techniques

1.1 Reminder (from e.g. [Len06]): Notations, Definitions and Basic Results

Definition 1 (Relations)

- We denote the composition of relations by $\cdot$, the identity relation by Id, and the inverse of a relation by $^{-1}$.
- If $\mathcal{D} \subseteq \mathcal{A}$, we write $\mathcal{R}(\mathcal{D})$ for $\{M \in \mathcal{B} | \ \exists N \in \mathcal{D}, N\mathcal{R}M\}$, or equivalently $\bigcup_{N \in \mathcal{D}}\{M \in \mathcal{B} | \ N\mathcal{R}M\}$. When $\mathcal{D}$ is the singleton $\{M\}$, we write $\mathcal{R}(M)$ for $\mathcal{R}(\{M\})$.
- We say that a relation $\mathcal{R} : \mathcal{A} \longrightarrow \mathcal{B}$ is *total* if $\mathcal{R}^{-1}(\mathcal{B}) = \mathcal{A}$.

Remark 1 Composition is associative, and identity relations are neutral for the composition operation.

Definition 2 (Reduction relation)

- A *reduction relation* on $\mathcal{A}$ is a relation from $\mathcal{A}$ to $\mathcal{A}$.
- Given a reduction relation $\rightarrow$ on $\mathcal{A}$, we define the set of $\rightarrow$-*reducible forms* (or just *reducible forms* when the relation is clear) as $\mathsf{rf}^{\rightarrow} := \{M \in \mathcal{A} |\ \exists N \in \mathcal{A}, M \rightarrow N\}$. We define the set of *normal forms* as $\mathsf{nf}^{\rightarrow} := \{M \in \mathcal{A} |\ \not\exists N \in \mathcal{A}, M \rightarrow N\}$.
- Given a reduction relation $\rightarrow$ on $\mathcal{A}$, we write $\leftarrow$ for $\rightarrow^{-1}$, and we define $\rightarrow^n$ by induction on the natural number n as follows:
  $\rightarrow^0 := \mathsf{Id}$
  $\rightarrow^{n+1} := \rightarrow \cdot \rightarrow^n (= \rightarrow^n \cdot \rightarrow)$
  $\rightarrow^+$ denotes the transitive closure of $\rightarrow$ (i.e. $\rightarrow^+ := \bigcup_{n\geq 1} \rightarrow^n$).
  $\rightarrow^*$ denotes the transitive and reflexive closure of $\rightarrow$ (i.e. $\rightarrow^* := \bigcup_{n\geq 0} \rightarrow^n$).
  $\leftrightarrow$ denotes the symmetric closure of $\rightarrow$ (i.e. $\leftrightarrow := \leftarrow \cup \rightarrow$).
  $\leftrightarrow^*$ denotes the transitive, reflexive and symmetric closure of $\rightarrow$.
- An *equivalence relation* on $\mathcal{A}$ is a transitive, reflexive and symmetric reduction relation on $\mathcal{A}$, i.e. a relation $\rightarrow = \leftrightarrow^*$, hence denoted more often by $\sim$, $\equiv$...
- Given a reduction relation $\rightarrow$ on $\mathcal{A}$ and a subset $\mathcal{B} \subseteq \mathcal{A}$, the *closure of $\mathcal{B}$ under $\rightarrow$* is $\rightarrow^*(\mathcal{B})$.

Definition 3 (Finitely branching relation) A reduction relation $\rightarrow$ on $\mathcal{A}$ is *finitely branching* if $\forall M \in \mathcal{A}$, $\rightarrow(M)$ is finite.

Definition 4 (Stability) Given a reduction relation $\rightarrow$ on $\mathcal{A}$, we say that a subset $\mathcal{T}$ of $\mathcal{A}$ is $\rightarrow$-*stable* (or *stable under* $\rightarrow$) if $\rightarrow(\mathcal{T}) \subseteq \mathcal{T}$.

Definition 5 (Strong simulation)
Let $\mathcal{R}$ be a relation between two sets $\mathcal{A}$ and $\mathcal{B}$, respectively equipped with the reduction relations $\rightarrow_{\mathcal{A}}$ and $\rightarrow_{\mathcal{B}}$.
$\rightarrow_{\mathcal{B}}$ *strongly simulates* $\rightarrow_{\mathcal{A}}$ *through* $\mathcal{R}$ if $(\mathcal{R}^{-1} \cdot \rightarrow_{\mathcal{A}}) \subseteq (\rightarrow^+_{\mathcal{B}} \cdot \mathcal{R}^{-1})$.

Remark 2

1. If $\rightarrow_{\mathcal{B}}$ strongly simulates $\rightarrow_{\mathcal{A}}$ through $\mathcal{R}$, and if $\rightarrow_{\mathcal{B}} \subseteq \rightarrow'_{\mathcal{B}}$ and $\rightarrow'_{\mathcal{A}} \subseteq \rightarrow_{\mathcal{A}}$, then $\rightarrow'_{\mathcal{B}}$ strongly simulates $\rightarrow'_{\mathcal{A}}$ through $\mathcal{R}$.
2. If $\rightarrow_{\mathcal{B}}$ strongly simulates $\rightarrow_{\mathcal{A}}$ and $\rightarrow'_{\mathcal{A}}$ through $\mathcal{R}$, then it also strongly simulates $\rightarrow_{\mathcal{A}} \cdot \rightarrow'_{\mathcal{A}}$ through $\mathcal{R}$.
3. Hence, if $\rightarrow_{\mathcal{B}}$ strongly simulates $\rightarrow_{\mathcal{A}}$ through $\mathcal{R}$, then it also strongly simulates $\rightarrow^+_{\mathcal{A}}$ through $\mathcal{R}$.

Definition 6 (Patriarchal) Given a reduction relation $\rightarrow$ on $\mathcal{A}$, we say that

- a subset $\mathcal{T}$ of $\mathcal{A}$ is $\rightarrow$-*patriarchal* (or just *patriarchal* when the relation is clear) if $\forall N \in \mathcal{A}$, $\rightarrow(N) \subseteq \mathcal{T} \Rightarrow N \in \mathcal{T}$.
- a predicate P on $\mathcal{A}$ is *patriarchal* if $\{M \in \mathcal{A} |\ P(M)\}$ is *patriarchal*.

Definition 7 (Normalising elements) Given a reduction relation $\rightarrow$ on $\mathcal{A}$, the set of $\rightarrow$-*strongly normalising* elements is

$$\mathsf{SN}^{\rightarrow} := \bigcap_{\mathcal{T} \text{ is patriarchal}} \mathcal{T}$$

Definition 8 (Bounded elements) The set of $\rightarrow$-*bounded* elements is defined as

$$\mathsf{BN}^{\rightarrow} := \bigcup_{n \geq 0} \mathsf{BN}_n^{\rightarrow}$$

where $\mathsf{BN}_n^{\rightarrow}$ is defined by induction on the natural number n as follows:

$$\begin{array}{lcl} \mathsf{BN}_0^{\rightarrow} & := & \mathsf{nf}^{\rightarrow} \\ \mathsf{BN}_{n+1}^{\rightarrow} & := & \{M \in \mathcal{A} |\ \exists n' \leq n,\ \rightarrow(M) \subseteq \mathsf{BN}_{n'}^{\rightarrow}\} \end{array}$$

Lemma 3 *If $\rightarrow$ is finitely branching, then $\mathsf{BN}^{\rightarrow}$ is patriarchal.*
As a consequence, $\mathsf{BN}^{\rightarrow} = \mathsf{SN}^{\rightarrow}$.

Lemma 4

1. *If $n < n'$ then $\mathsf{BN}_n^{\rightarrow} \subseteq \mathsf{BN}_{n'}^{\rightarrow} \subseteq \mathsf{BN}^{\rightarrow}$. In particular, $\mathsf{nf}^{\rightarrow} \subseteq \mathsf{BN}_n^{\rightarrow} \subseteq \mathsf{BN}^{\rightarrow}$.*
2. *$\mathsf{BN}^{\rightarrow} \subseteq \mathsf{SN}^{\rightarrow}$.*

Lemma 5

1. *$\mathsf{SN}^{\rightarrow}$ is patriarchal.*
2. *If $M \in \mathsf{BN}^{\rightarrow}$ then $\rightarrow(M) \subseteq \mathsf{BN}^{\rightarrow}$.*
   If $M \in \mathsf{SN}^{\rightarrow}$ then $\rightarrow(M) \subseteq \mathsf{SN}^{\rightarrow}$.

Theorem 6 (Induction principle) *Given a predicate P on $\mathcal{A}$,*
suppose $\forall M \in \mathsf{SN}^{\rightarrow}, (\forall N \in \rightarrow(M), P(N)) \Rightarrow P(M)$.
Then $\forall M \in \mathsf{SN}^{\rightarrow}, P(M)$.

When we use this theorem to prove a statement $P(M)$ for all M in $\mathsf{SN}^{\rightarrow}$, we just add $(\forall N \in \rightarrow(M), P(N))$ to the assumptions, which we call the induction hypothesis.

We say that we prove the statement by induction in $\mathsf{SN}^{\rightarrow}$.

Lemma 7

1. *If $\rightarrow_1 \subseteq \rightarrow_2$, then $\mathsf{nf}^{\rightarrow_1} \supseteq \mathsf{nf}^{\rightarrow_2}$, $\mathsf{SN}^{\rightarrow_1} \supseteq \mathsf{SN}^{\rightarrow_2}$,*
   and for all n, $\mathsf{BN}_n^{\rightarrow_1} \supseteq \mathsf{BN}_n^{\rightarrow_2}$.
2. *$\mathsf{nf}^{\rightarrow} = \mathsf{nf}^{\rightarrow^+}$, $\mathsf{SN}^{\rightarrow} = \mathsf{SN}^{\rightarrow^+}$, and for all n, $\mathsf{BN}_n^{\rightarrow^+} = \mathsf{BN}_n^{\rightarrow}$.*

Notice that this result enables us to use a stronger induction principle: in order to prove $\forall M \in \mathsf{SN}^{\rightarrow}, P(M)$, it now suffices to prove

$$\forall M \in \mathsf{SN}^{\rightarrow}, (\forall N \in \rightarrow^+(M), P(N)) \Rightarrow P(M)$$

This induction principle is called the *transitive induction in $\mathsf{SN}^{\rightarrow}$*.

Theorem 8 (Strong normalisation by strong simulation) *Let $\mathcal{R}$ be a relation between $\mathcal{A}$ and $\mathcal{B}$, equipped with the reduction relations $\rightarrow_{\mathcal{A}}$ and $\rightarrow_{\mathcal{B}}$.*

If $\rightarrow_{\mathcal{B}}$ strongly simulates $\rightarrow_{\mathcal{A}}$ through $\mathcal{R}$, then $\mathcal{R}^{-1}(\mathsf{SN}^{\rightarrow_{\mathcal{B}}}) \subseteq \mathsf{SN}^{\rightarrow_{\mathcal{A}}}$.

Lemma 9 *Given two reduction relations $\rightarrow_1$, $\rightarrow_2$, suppose that $\mathsf{SN}^{\rightarrow_1}$ is stable under $\rightarrow_2$. Then $\mathsf{SN}^{\rightarrow_1 \cup \rightarrow_2} = \mathsf{SN}^{\rightarrow_1^* \cdot \rightarrow_2} \cap \mathsf{SN}^{\rightarrow_1}$.*

1.2 A variant of adjournment for boundedness

Definition 9 Suppose $\to_{\mathcal{A}}$ is a reduction relation on $\mathcal{A}$, $\to_{\mathcal{B}}$ is a reduction relation on $\mathcal{B}$, $\mathcal{R}$ is a relation from $\mathcal{A}$ to $\mathcal{B}$.

$\to_{\mathcal{B}}$ *simulates the reduction lengths of* $\to_{\mathcal{A}}$ *through* $\mathcal{R}$ if

$$\forall k, \forall M, N \in \mathcal{A}, \forall P \in \mathcal{B},\ M \to^k_{\mathcal{A}} N \wedge M\mathcal{R}P \ \Rightarrow\ \exists Q \in \mathcal{B}, P \to^k_{\mathcal{B}} Q$$

Lemma 10 *Suppose* $\to_{\mathcal{A}}$ *is a reduction relation on* $\mathcal{A}$*,* $\to_{\mathcal{B}}$ *is a reduction relation on* $\mathcal{B}$*,* $\mathcal{R}$ *is a relation from* $\mathcal{A}$ *to* $\mathcal{B}$*.*

If $\to_{\mathcal{B}}$ *strongly simulates* $\to_{\mathcal{A}}$ *through* $\mathcal{R}$*, then* $\to_{\mathcal{B}}$ *simulates the reduction lengths of* $\to_{\mathcal{A}}$ *through* $\mathcal{R}$*.*

Proof: We prove by induction on k that $\forall k, \forall M, N \in \mathcal{A}^2, \forall P \in \mathcal{B},\ M \to^k_{\mathcal{A}} N \wedge M\mathcal{R}P \ \Rightarrow\ \exists Q, P \to^k_{\mathcal{B}} Q$.

- For $k = 0$: take $Q :=\ M = N$.
- Suppose it is true for k and take $M \to_{\mathcal{A}} M' \to^k_{\mathcal{A}} N$. The strong simulation gives P' such that $P \to^+_{\mathcal{B}} P'$ and $M'\mathcal{R}P'$. The induction hypothesis gives Q' such that $P' \to^k_{\mathcal{B}} Q'$. Then it suffices to take the prefix $P \to^{k+1}_{\mathcal{B}} Q$ (of length $k+1$) of $P \to^+_{\mathcal{B}} P' \to^k_{\mathcal{B}} Q'$.

□

Lemma 11 $\forall n, \forall M, \quad (\forall k, \forall N, M \to^k N \Rightarrow k \leq n) \quad \iff \quad M \in \mathsf{BN}^{\to}_n$

Proof: By transitive induction on n.

- For $n = 0$: clearly both sides are equivalent to $M \in \mathsf{nf}^{\to}$.
- Suppose it is true for all $i \leq n$.

  Suppose $\forall k, \forall N, M \to^k N \Rightarrow k \leq n+1$. Then take $M \to M'$ and assume $M' \to^{k'} N'$. We have $M \to^{k'+1} N'$ so from the hypothesis we derive $k'+1 \leq n+1$, i.e. $k' \leq n$. We apply the induction hypothesis on M' and get $M' \in \mathsf{BN}^{\to}_n$. By definition of $\mathsf{BN}^{\to}_{n+1}$ we get $M \in \mathsf{BN}^{\to}_{n+1}$.

  Conversely, suppose $M \in \mathsf{BN}^{\to}_{n+1}$ and $M \to^k N$. We must prove that $k \leq n+1$. If $k = 0$ we are done. If $k = k'+1$ we have $M \to M' \to^{k'} N$; by definition of $\mathsf{BN}^{\to}_{n+1}$ there is $i \leq n$ such that $M' \in \mathsf{BN}^{\to}_i$, and by induction hypothesis we have $k' \leq i$; hence $k = k'+1 \leq i+1 \leq n+1$.

□

Theorem 12 *Suppose* $\to_{\mathcal{A}}$ *is a reduction relation on* $\mathcal{A}$*,* $\to_{\mathcal{B}}$ *is a reduction relation on* $\mathcal{B}$*,* $\mathcal{R}$ *is a relation from* $\mathcal{A}$ *to* $\mathcal{B}$*.*

If $\to_{\mathcal{B}}$ *simulates the reduction lengths of* $\to_{\mathcal{A}}$ *through* $\mathcal{R}$*, then*

$$\forall n, \mathcal{R}^{-1}(\mathsf{BN}^{\to_{\mathcal{B}}}_n) \subseteq \mathsf{BN}^{\to_{\mathcal{A}}}_n \quad (\subseteq \mathsf{SN}^{\to_{\mathcal{A}}})$$

Proof: Suppose $N \in \mathsf{BN}^{\to_{\mathcal{B}}}_n$ and $M\mathcal{R}N$. If $M \to^k_{\mathcal{A}} M'$ then by simulation $N \to^k_{\mathcal{B}} N'$ so by Lemma 11 we have $k \leq n$. Hence by (the other direction of) Lemma 11 we have $M \in \mathsf{BN}^{\to_{\mathcal{A}}}_n$. □

Definition 10 Let $\to_1$ and $\to_2$ be two reduction relations on $\mathcal{A}$.
The relation $\to_1$ can be *strongly adjourned with respect to* $\to_2$ if whenever $M \to_1 N \to_2 P$ there exists Q such that $M \to_2 Q(\to_1 \cup \to_2)^+ P$.

Theorem 13 *Let $\to_1$ and $\to_2$ be two reduction relations on $\mathcal{A}$. If $\mathsf{nf}^{\to_2} \subseteq \mathsf{nf}^{\to_1}$ and $\to_1$ can be strongly adjourned with respect to $\to_2$ then $\mathsf{BN}^{\to_2} \subseteq \mathsf{BN}^{\to_1 \cup \to_2}$.*

Proof: From Theorem 12, it suffices to show that $\to_2$ simulates the reduction lengths of $\to_1 \cup \to_2$ through the identity. We show by induction on k that

$$\forall k, \forall M, N,\ M(\to_1 \cup \to_2)^k N \Rightarrow\ \exists Q, M \to_2^k Q$$

- For $k = 0$: take $Q :=\ M$
- For $k = 1$: If $M \to_2 N$ take $Q :=\ N$; if $M \to_1 N$ use the hypothesis $\mathsf{nf}^{\to_2} \subseteq \mathsf{nf}^{\to_1}$ to produce Q such that $M \to_2 Q$.
- Suppose it is true for $k+1$ and take $M(\to_1 \cup \to_2)P(\to_1 \cup \to_2)^{k+1}N$.

  The induction hypothesis provides T such that $P \to_2^{k+1} T$, in other words $P \to_2 S \to_2^k T$.

  If $M \to_2 P$ we are done. If $M \to_1 P$ we use the hypothesis of adjournment to transform $M \to_1 P \to_2 S$ into $M \to_2 P'(\to_1 \cup \to_2)^+ S$. Take the prefix $P'(\to_1 \cup \to_2)^{k+1} R$ (of length $k+1$) of $P'(\to_1 \cup \to_2)^+ S \to_2^k T$, and apply on this prefix the induction hypothesis to get $P' \to_2^{k+1} R$. We thus get $M \to_2^{k+2} R$.

□

2 Termination of λ-calculus with an extra call-by-value rule

We consider the following rule in λ-calculus:

$$\mathsf{assoc} \quad (\lambda x.M)\ ((\lambda y.N)\ P) \longrightarrow\ (\lambda y.(\lambda x.M)\ N)\ P)$$

We want to prove

Proposition 1 $\mathsf{SN}^{\beta} \subseteq \mathsf{SN}^{\mathsf{assoc}\beta}$.

Lemma 14 $\longrightarrow_{\mathsf{assoc}}$ *is terminating in λ-calculus.*

Proof: Each application of the rule decreases by one the number of pairs of λ that are not nested. □

To prove Proposition 1 above, it would thus be sufficient to prove that $\longrightarrow_{\mathsf{assoc}}$ could be adjourned with respect to $\longrightarrow_{\beta}$, in other words that $\longrightarrow_{\mathsf{assoc}} \cdot \longrightarrow_{\beta} \subseteq \longrightarrow_{\beta} \cdot \longrightarrow^*_{\mathsf{assoc}\beta}$ (the adjournment technique leads directly to the desired strong normalisation result). When trying to prove the property by induction and case analysis on the β-reduction following the assoc-reduction to be adjourned, all cases allow the adjournment but one, namely:

$$(\lambda x.M)\ ((\lambda y.N)\ P) \longrightarrow_{\mathsf{assoc}} (\lambda y.(\lambda x.M)\ N)\ P \longrightarrow_{\beta} (\lambda y.\left\{{}^{N}\!/\!_{x}\right\}M)\ P$$

Hence, we shall assume without loss of generality that the β-reduction is not of the above kind. For that we need to identify a sub-relation of β-reduction $\hookrightarrow$ such that

- $\longrightarrow_{\mathsf{assoc}}$ can now be adjourned with respect to $\hookrightarrow$
- we can justify that there is no loss of generality.

For this we give ourselves the possibility of marking λ-redexes and forbid reductions under their (marked) bindings, so that, if in the assoc-reduction above we make sure that $(\lambda y.(\lambda x.M)\ N)\ P)$ is marked, the problematic β-reduction is forbidden.

Hence we use the usual notation for a marked redex $(\overline{\lambda} y.Q)\ P$, but we can also see it as the construct $\mathsf{let}\ y = P\ \mathsf{in}\ Q$ of λ_{C} [Mog88] and other works on call-by-value λ-calculus. We start with a reminder about marked redexes.

Definition 11 The syntax of the λ-calculus is extended as follows:

$$M, N ::= x \mid \lambda x.M \mid M\ N \mid (\overline{\lambda} x.M)\ N$$

Reduction is given by the following system $\beta 12$:

$$\begin{array}{lll} \beta 1 & (\lambda x.M)\ N & \longrightarrow \left\{ {}^{M}\!/\!_{x} \right\} N \\ \beta 2 & (\overline{\lambda} x.M)\ N & \longrightarrow \left\{ {}^{M}\!/\!_{x} \right\} N \end{array}$$

The forgetful projection onto λ-calculus is straightforward:

$$\begin{array}{lll} \phi(x) & := & x \\ \phi(\lambda x.M) & := & \lambda x.\phi(M) \\ \phi(M\ N) & := & \phi(M)\ \phi(N) \\ \phi((\overline{\lambda} x.M)\ N) & := & (\lambda x.\phi(M))\ \phi(N) \end{array}$$

Remark 15 Clearly, $\longrightarrow_{\beta 12}$ strongly simulates $\longrightarrow_{\beta}$ through ϕ^{-1} and $\longrightarrow_{\beta}$ strongly simulates $\longrightarrow_{\beta 12}$ through ϕ.

2.1 Reducing under $\overline{\lambda}$ and erasing $\overline{\lambda}$ can be strongly adjourned

In this section we identify the reduction notion $\hookrightarrow$ $(\subseteq \longrightarrow_{\beta 12})$ and we argue against the loss of generality by proving that $\longrightarrow_{\beta 12}$ can be strongly adjourned with respect to $\hookrightarrow$.

We thus split the reduction system $\beta 12$ into two cases depending on whether or not a reduction throws away an argument that contains some markings:

Definition 12

$$\begin{array}{ll} \beta\kappa & \left\{ \begin{array}{lll} (\lambda x.M)\ P & \longrightarrow M & \text{if } x \notin \mathsf{FV}(M) \text{ and there is a term } (\overline{\lambda} x.N)\ Q \sqsubseteq P \\ (\overline{\lambda} x.M)\ P & \longrightarrow M & \text{if } x \notin \mathsf{FV}(M) \text{ and there is a term } (\overline{\lambda} x.N)\ Q \sqsubseteq P \end{array} \right. \\ \beta\overline{\kappa} & \left\{ \begin{array}{lll} (\lambda x.M)\ P & \longrightarrow M & \text{if } x \in \mathsf{FV}(M) \text{ or there is no term } (\overline{\lambda} x.N)\ Q \sqsubseteq P \\ (\overline{\lambda} x.M)\ P & \longrightarrow M & \text{if } x \in \mathsf{FV}(M) \text{ or there is no term } (\overline{\lambda} x.N)\ Q \sqsubseteq P \end{array} \right. \end{array}$$

Remark 16 Clearly, $\longrightarrow_{\beta 12} = \longrightarrow_{\beta\kappa} \cup \longrightarrow_{\beta\overline{\kappa}}$.

No we distinguish whether or not a reduction occurs underneath a marked redex, via the following rule and the following notion of contextual closure:

Definition 13

$$\overline{\beta} \quad (\overline{\lambda} x.M)\ P \quad \longrightarrow \quad (\overline{\lambda} x.N)\ P \quad \text{if } M \longrightarrow_{\beta 12} N$$

Now we define a weak notion of contextual closure for a rewriting system i:

$$\frac{i:\ M \longrightarrow\ N}{M \rightharpoonup_i N} \qquad \frac{M \rightharpoonup_i N}{\lambda x.M \rightharpoonup_i \lambda x.N} \qquad \frac{M \rightharpoonup_i N}{M\ P \rightharpoonup_i N\ P} \qquad \frac{M \rightharpoonup_i N}{P\ M \rightharpoonup_i P\ N}$$

$$\frac{M \rightharpoonup_i N}{(\overline{\lambda} x.P)\ M \rightharpoonup_i (\overline{\lambda} x.P)\ N}$$

Finally we use the following abbreviations:

Definition 14 Let $\hookrightarrow := \rightharpoonup_{\beta\overline{\kappa}}$ and $\rightsquigarrow_1 := \rightharpoonup_{\beta\kappa}$ and $\rightsquigarrow_2 := \rightharpoonup_{\overline{\beta}}$.

Remark 17 Clearly, $\longrightarrow_{\beta 12} = \hookrightarrow \cup \rightsquigarrow_1 \cup \rightsquigarrow_2$.

Lemma 18 *If $(\overline{\lambda} x.N)\ Q \sqsubseteq P$, then there is P' such that $P \hookrightarrow P'$.*

Proof: By induction on P

- The case $P = y$ is vacuous.
- For $P = \lambda y.M$, we have $(\overline{\lambda} x.N)\ Q \sqsubseteq M$ and the induction hypothesis provides $M \hookrightarrow M'$, so $\lambda y.M \hookrightarrow \lambda y.M'$.
- For $P = M_1\ M_2$, we have either $(\overline{\lambda} x.N)\ Q \sqsubseteq M_1$ or $(\overline{\lambda} x.N)\ Q \sqsubseteq M_2$. In the former case the induction hypothesis provides $M_1 \hookrightarrow M_1'$, so $M_1\ M_2 \hookrightarrow M_1'\ M_2$. The latter case is similar.
- Suppose $P = (\overline{\lambda} y.M_1)\ M_2$. If there is a term $(\overline{\lambda} x'.N')\ Q' \sqsubseteq M_2$, the induction hypothesis provides $M_2 \hookrightarrow M_2'$, so $(\overline{\lambda} y.M_1)\ M_2 \hookrightarrow (\overline{\lambda} y.M_1)\ M_2'$. If there is no such term $(\overline{\lambda} x'.N')\ Q' \sqsubseteq M_2$, we have $(\overline{\lambda} y.M_1)\ M_2 \hookrightarrow \left\{{}^{M_2}\!/_{y}\right\} M_1$.

□

Lemma 19 $\rightsquigarrow_1 \subseteq \hookrightarrow \cdot \rightsquigarrow_1$

Proof: By induction on the reduction step $\rightsquigarrow_1$.

For the base cases $(\lambda x.M)\ P \longrightarrow_{\beta\kappa}\ M$ or $(\overline{\lambda} x.M)\ P \longrightarrow_{\beta\kappa}\ M$ with $x \notin \mathsf{FV}(M)$ and $(\overline{\lambda} y.N)\ Q \sqsubseteq P$, Lemma 18 provides the reduction $P \hookrightarrow P'$, so $(\lambda x.M)\ P \hookrightarrow (\lambda x.M)\ P' \rightsquigarrow_1 M$ and $(\overline{\lambda} x.M)\ P \hookrightarrow (\overline{\lambda} x.M)\ P' \rightsquigarrow_1 M$.

The induction step is straightforward as the same contextual closure is used on both sides (namely, the weak one). □

Lemma 20 $\rightsquigarrow_2 \cdot \hookrightarrow\ \subseteq\ \hookrightarrow \cdot \longrightarrow^{+}_{\beta 12}$

Proof: By induction on the reduction step $\hookrightarrow$.

- For the base case where the $\beta\overline{\kappa}$-reduction is a $\beta 2$-reduction, we have $M \rightsquigarrow_2 (\overline{\lambda} x.N)\ P \hookrightarrow \left\{{}^{P}\!/_{x}\right\} N$ with $x \in \mathsf{FV}(N)$ or P has no marked redex as a subterm. We do a case analysis on the reduction step $M \rightsquigarrow_2 (\overline{\lambda} x.N)\ P$.

  If $M = (\overline{\lambda} x.N')\ P \rightsquigarrow_2 (\overline{\lambda} x.N)\ P$ because $N' \longrightarrow_{\beta 12}\ N$ then $(\overline{\lambda} x.N')\ P \hookrightarrow \left\{{}^{P}\!/_{x}\right\} N' \longrightarrow_{\beta 12}\ \left\{{}^{P}\!/_{x}\right\} N$.

  If $M = (\overline{\lambda} x.N)\ P' \rightsquigarrow_2 (\overline{\lambda} x.N)\ P$ because $P' \rightsquigarrow_2 P$, then it means that P has a marked redex as a subterm, so we must have $x \in \mathsf{FV}(N)$. Hence $(\overline{\lambda} x.N)\ P' \hookrightarrow \left\{{}^{P'}\!/_{x}\right\} N \longrightarrow^{+}_{\beta 12}\ \left\{{}^{P}\!/_{x}\right\} N$.

- For the base case where the $\beta\overline{\kappa}$-reduction is a $\beta 1$-reduction, we have $M \rightsquigarrow_2 (\lambda x.N)\ P \hookrightarrow \left\{ {}^{P}\!/\!_{x} \right\} N$ with $x \in \mathsf{FV}(N)$ or P has no marked redex as a subterm. We do a case analysis on the reduction step $M \rightsquigarrow_2 (\lambda x.N)\ P$.

  If $M = M'\ P \rightsquigarrow_2 (\lambda x.N)\ P$ because $M' \rightsquigarrow_2 \lambda x.N$ then M' must be of the form $\lambda x.M''$ with $M'' \rightsquigarrow_2 N$. Then $(\lambda x.M'')\ P \hookrightarrow \left\{ {}^{P}\!/\!_{x} \right\} M''$ (in case P has a marked subterm, notice that $x \in \mathsf{FV}(N) \subseteq \mathsf{FV}(M'')$), and $\left\{ {}^{P}\!/\!_{x} \right\} M'' \longrightarrow_{\beta 12} \left\{ {}^{P}\!/\!_{x} \right\} N$.

  If $M = (\lambda x.N)\ P' \rightsquigarrow_2 (\lambda x.N)\ P$ because $P' \rightsquigarrow_2 P$, then it means that P has a marked redex as a subterm, so we must have $x \in \mathsf{FV}(N)$. Hence $(\lambda x.N)\ P' \hookrightarrow \left\{ {}^{P'}\!/\!_{x} \right\} N \longrightarrow^+_{\beta 12} \left\{ {}^{P}\!/\!_{x} \right\} N$.

- The closure under λ is straightforward.

- For the closure under application, left-hand side, we have $M \rightsquigarrow_2 N\ P \hookrightarrow N'\ P$ with $N \hookrightarrow N'$. We do a case analysis on the reduction step $M \rightsquigarrow_2 N\ P$.

  If $M = M'\ P \rightsquigarrow_2 N\ P$ with $M' \rightsquigarrow_2 N$, the induction hypothesis gives $M' \hookrightarrow \cdot \longrightarrow^+_{\beta 12} N'$ and the weak contextual closure gives $M'\ P \hookrightarrow \cdot \longrightarrow^+_{\beta 12} N'\ P$.

  If $M = N\ P' \rightsquigarrow_2 N\ P$ with $P' \rightsquigarrow_2 P$, we can also derive $N\ P' \hookrightarrow N'\ P' \longrightarrow_{\beta 12} N'\ P$.

- For the closure under application, right-hand side, we have $M \rightsquigarrow_2 N\ P \hookrightarrow N\ P'$ with $P \hookrightarrow P'$. We do a case analysis on the reduction step $M \rightsquigarrow_2 N\ P$.

  If $M = M'\ P \rightsquigarrow_2 N\ P$ with $M' \rightsquigarrow_2 N$, we can also derive $M'\ P \hookrightarrow M'\ P' \longrightarrow_{\beta 12} N\ P'$.

  If $M = N\ M' \rightsquigarrow_2 N\ P$ with $M' \rightsquigarrow_2 P$, the induction hypothesis gives $M' \hookrightarrow \cdot \longrightarrow^+_{\beta 12} P'$ and the weak contextual closure gives $N\ M' \hookrightarrow \cdot \longrightarrow^+_{\beta 12} N\ P'$.

- For the closure under marked redex we have $M \rightsquigarrow_2 (\overline{\lambda} x.P)\ N \hookrightarrow (\overline{\lambda} x.P)\ N'$ with $N \hookrightarrow N'$. We do a case analysis on the reduction step $M \rightsquigarrow_2 (\overline{\lambda} x.P)\ N$.

  If $M = (\overline{\lambda} x.P')\ N \rightsquigarrow_2 (\overline{\lambda} x.P)\ N$ because $P' \longrightarrow_{\beta 12} P$, we can also derive $(\overline{\lambda} x.P')\ N \hookrightarrow (\overline{\lambda} x.P')\ N' \longrightarrow_{\beta 12} (\overline{\lambda} x.P)\ N'$.

  If $M = (\overline{\lambda} x.P)\ M' \rightsquigarrow_2 (\overline{\lambda} x.P)\ N$ with $M' \rightsquigarrow_2 N$, the induction hypothesis gives $M' \hookrightarrow Q \longrightarrow^+_{\beta 12} N'$ and the weak contextual closure gives $(\overline{\lambda} x.P)\ M' \hookrightarrow (\overline{\lambda} x.P)\ Q \longrightarrow^+_{\beta 12} (\overline{\lambda} x.P)\ N'$.

□

Corollary 21 $\longrightarrow_{\beta 12}$ *can be strongly adjourned with respect to* $\hookrightarrow$.

Proof: Straightforward from the last two theorems, and Remark 17. □

2.2 assoc-reduction

We introduce two new rules in the marked λ-calculus to simulate assoc:

$$\begin{array}{lll} \overline{\mathsf{assoc}} & (\overline{\lambda} x.M)\ (\overline{\lambda} y.N)\ P & \longrightarrow\ (\overline{\lambda} y.(\overline{\lambda} x.M)\ N)\ P \\ \mathsf{act} & (\lambda x.M)\ N & \longrightarrow\ (\overline{\lambda} x.M)\ N \end{array}$$

Remark 22 Clearly, $\longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}}$ strongly simulates $\longrightarrow_{\mathsf{assoc}}$ through ϕ^{-1}.

Notice that with the $\mathsf{let} = \mathsf{in}$ -notation, $\overline{\mathsf{assoc}}$ and act are simply the rules of λ_{C}

$$\begin{array}{lll} \overline{\mathsf{assoc}} & \mathsf{let}\ x = (\mathsf{let}\ y = P\ \mathsf{in}\ N)\ \mathsf{in}\ M & \longrightarrow \mathsf{let}\ y = P\ \mathsf{in}\ \mathsf{let}\ x = N\ \mathsf{in}\ M \\ \mathsf{act} & (\lambda x.M)\ N & \longrightarrow \mathsf{let}\ x = N\ \mathsf{in}\ M \end{array}$$

Lemma 23 $\longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} \cdot \hookrightarrow \subseteq \hookrightarrow \cdot \longrightarrow^*_{\overline{\mathsf{assoc}}\mathsf{act}}$

Proof: By induction on the reduction step $\hookrightarrow$.

- For the first base case, we have $M \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} (\lambda x.N)\ P \hookrightarrow \left\{{}^{P}\!/\!_{x}\right\}N$ with $x \in \mathsf{FV}(N)$ or P has no marked subterm. Since root $\overline{\mathsf{assoc}}\mathsf{act}$-reduction produces neither λ-abstractions nor applications at the root, note that M has to be of the form $(\lambda x.N')\ P'$, with either $N' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} N$ (and $P' = P$) or $P' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} P$ (and $N' = N$). In both cases, $x \in \mathsf{FV}(N) \subseteq \mathsf{FV}(N')$ or P' has no marked subterm, so we also have $(\lambda x.N')\ P' \hookrightarrow \left\{{}^{P'}\!/\!_{x}\right\}N' \longrightarrow^*_{\overline{\mathsf{assoc}}\mathsf{act}} \left\{{}^{P}\!/\!_{x}\right\}N$.

- For the second base case, we have $M \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} (\overline{\lambda}x.N)\ P \hookrightarrow \left\{{}^{P}\!/\!_{x}\right\}N$ with $x \in \mathsf{FV}(N)$ or P has no marked subterm. We do a case analysis on $M \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} (\overline{\lambda}x.N)\ P$.

  If $M = (\overline{\lambda}x'.M_1)\ (\overline{\lambda}x.M_2)\ P \longrightarrow_{\overline{\mathsf{assoc}}} (\overline{\lambda}x.(\overline{\lambda}x'.M_1)\ M_2)\ P$ with $N = (\overline{\lambda}x'.M_1)\ M_2$, we also have $M = (\overline{\lambda}x'.M_1)\ (\overline{\lambda}x.M_2)\ P \hookrightarrow (\overline{\lambda}x'.M_1)\ \left\{{}^{P}\!/\!_{x}\right\}M_2 = \left\{{}^{P}\!/\!_{x}\right\}N$.

  If $M = (\lambda x.N)\ P \longrightarrow_{\mathsf{act}} (\overline{\lambda}x.N)\ P$ then $M \hookrightarrow \left\{{}^{P}\!/\!_{x}\right\}N$.

  If $M = (\overline{\lambda}x.N')\ P' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} (\overline{\lambda}x.N)\ P$ with either $N' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} N$ (and $P' = P$) or $P' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} P$ (and $N' = N$), we have, in both cases, $x \in \mathsf{FV}(N) \subseteq \mathsf{FV}(N')$ or P' has no marked subterm, so we also have $(\lambda x.N')\ P' \hookrightarrow \left\{{}^{P'}\!/\!_{x}\right\}N' \longrightarrow^*_{\overline{\mathsf{assoc}}\mathsf{act}} \left\{{}^{P}\!/\!_{x}\right\}N$.

- The closure under λ is straightforward.

- For the closure under application, left-hand side, we have $Q \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M\ N \hookrightarrow M'\ N$ with $M \hookrightarrow M'$. We do a case analysis on $Q \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M\ N$.

  If $Q = M''\ N \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M\ N$ with $M'' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M$, the induction hypothesis provides $M'' \hookrightarrow \cdot \longrightarrow^*_{\overline{\mathsf{assoc}}\mathsf{act}} M'$ so $M''\ N \hookrightarrow \cdot \longrightarrow^*_{\overline{\mathsf{assoc}}\mathsf{act}} M'\ N$.

  If $Q = M\ N' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M\ N$ with $N' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} N$, we also have $M\ N' \hookrightarrow M'\ N' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M'\ N$.

- For the closure under application, right-hand side, we have $Q \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M\ N \hookrightarrow M\ N'$ with $N \hookrightarrow N'$. We do a case analysis on $Q \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M\ N$.

  If $Q = M'\ N \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M\ N$ with $M' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M$, we also have $M'\ N \hookrightarrow M'\ N' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M\ N'$.

  If $Q = M\ N'' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} M\ N$ with $N'' \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} N$, the induction hypothesis provides $N'' \hookrightarrow \cdot \longrightarrow^*_{\overline{\mathsf{assoc}}\mathsf{act}} N'$ so $M\ N'' \hookrightarrow \cdot \longrightarrow^*_{\overline{\mathsf{assoc}}\mathsf{act}} M\ N'$.

- For the closure under marked redex, the $\hookrightarrow$-reduction can only come from the right-hand side because of the weak contextual closure ($\hookrightarrow$ does not reduce under $\overline{\lambda}$), so we have $Q \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} (\overline{\lambda}y.M)\ P \hookrightarrow (\overline{\lambda}y.M)\ P'$ with $P \hookrightarrow P'$. We do a case analysis on $Q \longrightarrow_{\overline{\mathsf{assoc}}\mathsf{act}} (\overline{\lambda}y.M)\ P$.

  If $Q = (\overline{\lambda}x.M')\ (\overline{\lambda}y.N)\ P \longrightarrow_{\overline{\mathsf{assoc}}} (\overline{\lambda}y.(\overline{\lambda}x.M')\ N)\ P$ with $M = (\overline{\lambda}x.M')\ N$, we also have $Q = (\overline{\lambda}x.M')\ (\overline{\lambda}y.N)\ P \hookrightarrow (\overline{\lambda}x.M')\ (\overline{\lambda}y.N)\ P' \longrightarrow_{\overline{\mathsf{assoc}}} (\overline{\lambda}y.(\overline{\lambda}x.M')\ N)\ P'$.

  If $Q = (\lambda y.M)\ P \longrightarrow_{\mathsf{act}} (\overline{\lambda}y.M)\ P$, then we also have $Q = (\lambda y.M)\ P \hookrightarrow (\lambda y.M)\ P' \longrightarrow_{\mathsf{act}} (\overline{\lambda}y.M)\ P'$.

□

Lemma 24 $\longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}$ *can be strongly adjourned with respect to* $\hookrightarrow$*.*

Proof: We prove that $\forall k, \longrightarrow^k_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12} \cdot \hookrightarrow \subseteq \hookrightarrow \cdot \longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}$ by induction on k.

- For $k = 0$, this is Corollary 21.
- Suppose it is true for k. By the induction hypthesis we get

  $$\longrightarrow_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow^k_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12} \cdot \hookrightarrow \subseteq \longrightarrow_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \hookrightarrow \cdot \longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}$$

  Then by Lemma 23 we get

  $$\longrightarrow_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \hookrightarrow \cdot \longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12} \subseteq \hookrightarrow \cdot \longrightarrow_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}$$

□

Remark 25 Note from Lemma 18 that $\mathsf{nf}^{\hookrightarrow} \subseteq \mathsf{nf}^{\leadsto_1 \cup \leadsto_2} \subseteq \mathsf{nf}^{\longrightarrow_{\beta 12}} \subseteq \mathsf{nf}^{\longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}}$.

Theorem 26 $\mathsf{BN}^{\hookrightarrow} \subseteq \mathsf{BN}^{\longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}}$

Proof: We apply Theorem 13, since $\mathsf{nf}^{\hookrightarrow} \subseteq \mathsf{nf}^{\longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}}$ and clearly

$$(\longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}) \cup \hookrightarrow = \longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}$$

□

Theorem 27 $\mathsf{BN}^{\beta} \subseteq \mathsf{BN}^{\longrightarrow^*_{\mathsf{assoc}} \cdot \longrightarrow_{\beta}}$

Proof: Since $\longrightarrow_\beta$ strongly simulates $\hookrightarrow$ through ϕ, we have $\phi^{-1}(\mathsf{BN}^{\beta}) \subseteq \mathsf{BN}^{\hookrightarrow} \subseteq \mathsf{BN}^{\longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}}$. Hence $\phi(\phi^{-1}(\mathsf{BN}^{\beta})) \subseteq \phi(\mathsf{BN}^{\longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}})$. Since ϕ is surjective, $\mathsf{BN}^{\beta} = \phi(\phi^{-1}(\mathsf{BN}^{\beta}))$. Hence $\mathsf{BN}^{\beta} \subseteq \phi(\mathsf{BN}^{\longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}})$. Also, $\longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}$ strongly simulates $\longrightarrow^*_{\mathsf{assoc}} \cdot \longrightarrow_\beta$ through ϕ^{-1}, so $\phi(\mathsf{BN}^{\longrightarrow^*_{\overline{\mathsf{assoc}},\mathsf{act}} \cdot \longrightarrow_{\beta 12}}) \subseteq \mathsf{BN}^{\longrightarrow^*_{\mathsf{assoc}} \cdot \longrightarrow_{\beta}}$. □

Theorem 28 $\mathsf{SN}^{\beta} \subseteq \mathsf{SN}^{\mathsf{assoc}\beta}$

Proof: First, from Lemma 4, $\mathsf{BN}^{\longrightarrow^*_{\mathsf{assoc}} \cdot \longrightarrow_{\beta}} \subseteq \mathsf{SN}^{\longrightarrow^*_{\mathsf{assoc}} \cdot \longrightarrow_{\beta}}$. Then from Lemma 14, $\longrightarrow_{\mathsf{assoc}}$ is terminating and hence $\mathsf{SN}^{\mathsf{assoc}}$ is stable under $\longrightarrow_\beta$. Hence we can apply Lemma 9 to get $\mathsf{SN}^{\mathsf{assoc}\beta} = \mathsf{SN}^{\longrightarrow^*_{\mathsf{assoc}} \cdot \longrightarrow_{\beta}}$. From the previous theorem we thus have $\mathsf{BN}^{\beta} \subseteq \mathsf{SN}^{\mathsf{assoc}\beta}$. Now, noticing that β-reduction in λ-calculus is finitely branching, Lemma 3 gives $\mathsf{BN}^{\beta} = \mathsf{SN}^{\beta}$ and thus $\mathsf{SN}^{\beta} \subseteq \mathsf{SN}^{\mathsf{assoc}\beta}$. □

3 λLJQ

Definition 15 (λLJQ) • The syntax of λLJQ comes as two syntactic categories, the first one of which being that of *values*:

$$\begin{array}{lll} V, V' & ::= & x \mid \lambda x.M \mid \langle V \nwarrow x.V' \rangle \\ M, N, P & ::= & [V] \mid x[V, y.N] \mid \langle V \nwarrow x.N \rangle \mid \langle M \dagger x.N \rangle \end{array}$$

- We define *x-covalues* as those terms of the form $[x]$ or $x[V, y.M]$ with $x \notin \mathsf{FV}(V) \cup \mathsf{FV}(M)$.
- We define *principal cuts* as those terms of the form $\langle [V] \dagger x.M \rangle$ where M is an x-covalue.

$$\frac{}{\Gamma, x{:}A \vdash^{\mathsf{V}} x{:}A} \qquad \frac{\Gamma \vdash^{\mathsf{V}} V{:}A}{\Gamma \vdash [V]{:}A}$$

$$\frac{\Gamma, x{:}A \vdash M{:}B}{\Gamma \vdash^{\mathsf{V}} \lambda x.M{:}A{\to}B} \qquad \frac{\Gamma, x{:}A{\to}B \vdash^{\mathsf{V}} V{:}A \quad \Gamma, x{:}A{\to}B, y{:}B \vdash N{:}C}{\Gamma, x{:}A{\to}B \vdash x[V, y.N]{:}C}$$

$$\frac{\Gamma \vdash^{\mathsf{V}} V{:}A \quad \Gamma, x{:}A \vdash^{\mathsf{V}} V'{:}B}{\Gamma \vdash^{\mathsf{V}} \left\langle V \diagdown x.V' \right\rangle{:}B} \qquad \frac{\Gamma \vdash^{\mathsf{V}} V{:}A \quad \Gamma, x{:}A \vdash N{:}B}{\Gamma \vdash \left\langle V \diagdown x.N \right\rangle{:}B}$$

$$\frac{\Gamma \vdash M{:}A \quad \Gamma, x{:}A \vdash N{:}B}{\Gamma \vdash \langle M \dagger x.N \rangle{:}B}$$

Figure 1: Typing system of $\lambda\mathsf{LJQ}$

- The typing rules are shown in Fig. 1. Derivability, in the typing system of $\lambda\mathsf{LJQ}$, of the sequents $\Gamma \vdash^{\mathsf{V}} V{:}A$ and $\Gamma \vdash M{:}A$, is denoted $\Gamma \vdash^{\mathsf{V}}_{\lambda\mathsf{LJQ}} V{:}A$ and $\Gamma \vdash_{\lambda\mathsf{LJQ}} M{:}A$, respectively.
- The reduction system, also called $\lambda\mathsf{LJQ}$, is shown in Fig. 2.

$$\begin{array}{rcl}
\langle [\lambda x.M] \dagger y.y[V, z.P] \rangle & \longrightarrow & \langle \langle [V] \dagger x.M \rangle \dagger z.P \rangle \quad \text{if } y \notin \mathsf{FV}(V) \cup \mathsf{FV}(P) \\
\langle [x] \dagger y.N \rangle & \longrightarrow & \{{}^{x}\!/\!{}_{y}\}N \\
\langle M \dagger y.[y] \rangle & \longrightarrow & M
\end{array}$$

$$\begin{array}{rcl}
\langle z[V, y.P] \dagger x.N \rangle & \longrightarrow & z[V, y.\, \langle P \dagger x.N \rangle] \\
\langle \langle [V'] \dagger y.y[V, z.P] \rangle \dagger x.N \rangle & \longrightarrow & \langle [V'] \dagger y.y[V, z.\, \langle P \dagger x.N \rangle] \rangle \\
 & & \qquad \text{if } y \notin \mathsf{FV}(V) \cup \mathsf{FV}(P) \\
\langle \langle M \dagger y.P \rangle \dagger x.N \rangle & \longrightarrow & \langle M \dagger y.\langle P \dagger x.N \rangle \rangle \\
 & & \quad \text{if the redex is not one of the previous rule} \\
\langle [\lambda y.M] \dagger x.N \rangle & \longrightarrow & \left\langle \lambda y.M \diagdown x.N \right\rangle \text{ if } N \text{ is not an } x\text{-covalue}
\end{array}$$

$$\begin{array}{lcl}
\left\langle V \diagdown x.x \right\rangle & \longrightarrow & V \\
\left\langle V \diagdown x.y \right\rangle & \longrightarrow & y \\
\left\langle V \diagdown x.\lambda y.M \right\rangle & \longrightarrow & \lambda y.\, \left\langle V \diagdown x.M \right\rangle
\end{array}$$

$$\begin{array}{lcl}
\left\langle V \diagdown x.[V'] \right\rangle & \longrightarrow & [\left\langle V \diagdown x.V' \right\rangle] \\
\left\langle V \diagdown x.x[V', z.P] \right\rangle & \longrightarrow & \left\langle [V] \dagger x.x[\left\langle V \diagdown x.V' \right\rangle, z.\, \left\langle V \diagdown x.P \right\rangle] \right\rangle \\
\left\langle V \diagdown x.x'[V', z.P] \right\rangle & \longrightarrow & x'[\left\langle V \diagdown x.V' \right\rangle, z.\, \left\langle V \diagdown x.P \right\rangle] \\
\left\langle V \diagdown x.\langle M \dagger y.P \rangle \right\rangle & \longrightarrow & \left\langle \left\langle V \diagdown x.M \right\rangle \dagger y.\left\langle V \diagdown x.P \right\rangle \right\rangle
\end{array}$$

Figure 2: Reduction rules of $\lambda\mathsf{LJQ}$

As mentioned earlier, the purpose of this paper is to prove that if $\Gamma \vdash M : A$ then $M \in \mathsf{SN}$.

$$\frac{}{[V] \in \mathcal{V}} \qquad \frac{M \in \mathcal{V}}{\langle V \nwarrow y.M \rangle \in \mathcal{V}}$$

Figure 3: Pseudo values

$$\frac{}{x[V, y.M] \in \mathcal{S}_x}\ x \notin \mathsf{FV}(V) \cup \mathsf{FV}(M)$$

$$\frac{M \in \mathcal{S}_x}{\langle V \nwarrow y.M \rangle \in \mathcal{S}_x}\ x \notin \mathsf{FV}(V) \qquad \frac{M \in \mathcal{S}_x}{\langle M \dagger y.N \rangle \in \mathcal{S}_x}\ x \notin \mathsf{FV}(N)$$

Figure 4: Pseudo x-covalues

4 Strong normalisation of λLJQ

4.1 An encoding into λ-calculus

Definition 16 In Fig. 3 and Fig. 4, we inductively define the set $\mathcal{V}$ of *pseudo values* and the set $\mathcal{S}_x$ of *pseudo x-covalues*, respectively. A cut $\langle N \dagger x.M \rangle$ is said to be *pseudo principal* if $N \in \mathcal{V}$ and $M \in \mathcal{S}_x$.

Remark 29 Note that $\mathcal{V} \cap \mathcal{S}_x = \emptyset$ and its immediate corollary that pseudo principal cuts cannot be in $\mathcal{S}_x$ (nor of course in $\mathcal{V}$).

Remark 30 $\mathcal{V}$ and $\mathcal{S}_x$ are stable under reduction.

Remark 31 Suppose $x \neq y$ and $x \neq z$. If M is in $\mathcal{S}_x$ (resp. in $\mathcal{V}$) then so is $\{{}^{z}\!/\!_{y}\}M$. Hence if M is a pseudo principal cut then so is $\{{}^{z}\!/\!_{y}\}M$.

Definition 17

- We give in Fig. 5 an encoding of λLJQ into the λ-calculus.

$$\begin{array}{lcll}
\overline{x} & := & x & \\
\overline{\lambda x.M} & := & \mathsf{I}\ \lambda x.\overline{M} & \\
\overline{\langle V \nwarrow x.V' \rangle} & := & \left\{{}^{\overline{V}}\!/\!_{x}\right\}\overline{V'} & \\
\hline
\overline{[V]} & := & \overline{V} & \\
\overline{x[V, y.M]} & := & (\lambda y.\overline{M})\ (x\ \overline{V}) & \\
\overline{\langle N \dagger x.M \rangle} & := & \left\{{}^{\overline{N}}\!/\!_{x}\right\}\overline{M} & N \in \mathcal{V} \wedge M \in \mathcal{S}_x \\
\overline{\langle N \dagger x.M \rangle} & := & (\lambda x.\overline{M})\ (\overline{N}) & \text{otherwise} \\
\overline{\langle V \nwarrow x.M \rangle} & := & \left\{{}^{\overline{V}}\!/\!_{x}\right\}\overline{M} &
\end{array}$$

Figure 5: From λLJQ to λ-calculus

- We define in λLJQ the set of *β-bounded* terms (and values) as
$\mathcal{B} := \{M \in \lambda\mathsf{LJQ} \mid \forall N \sqsubseteq M, \overline{N} \in \mathsf{SN}^{\beta} \wedge \forall V' \sqsubseteq M, \overline{V'} \in \mathsf{SN}^{\beta}\}$
$\cup \{V \in \lambda\mathsf{LJQ} \mid \forall N \sqsubseteq V, \overline{N} \in \mathsf{SN}^{\beta} \wedge \forall V' \sqsubseteq V, \overline{V}' \in \mathsf{SN}^{\beta}\}$

Remark 32 Note that $\mathsf{FV}(\overline{M}) \subseteq \mathsf{FV}(M)$ and, from Remark 31, $\overline{\{{}^{y}\!/\!_{x}\}M} = \{{}^{y}\!/\!_{x}\}\overline{M}$.

Remark 33 $(\lambda x.\overline{M})\ \overline{N} \longrightarrow^*_{\mathsf{assoc}\beta} \overline{\langle N \dagger x.M\rangle} \longrightarrow^*_{\mathsf{assoc}\beta} \left\{{}^{\overline{N}}\!/\!_{x}\right\}\overline{M}$

Theorem 34 *If $M \in \mathcal{B}$ (resp. $V \in \mathcal{B}$) and $M \longrightarrow_{\lambda\mathsf{LJQ}} N$ (resp. $V \longrightarrow_{\lambda\mathsf{LJQ}} V'$), then $N \in \mathcal{B}$ (resp. $V' \in \mathcal{B}$) and $\overline{M} \longrightarrow^*_{\mathsf{assoc}\beta} \overline{N}$ (resp. $\overline{V} \longrightarrow^*_{\mathsf{assoc}\beta} \overline{V'}$).*

Proof: By induction on the reduction steps $M \longrightarrow_{\lambda\mathsf{LJQ}} N$ (resp. $V \longrightarrow_{\lambda\mathsf{LJQ}} V'$). The following table gives the base case of the induction: for each reduction rule, the left-hand side shows a term to which $\overline{M}$ (resp. $\overline{V}$) $\mathsf{assoc}\beta$-reduces (by Remark 33), while the right-hand side shows a term that $\mathsf{assoc}\beta$-reduces (by Remark 33) to $\overline{N}$ (resp. $\overline{V'}$).

$$
\begin{array}{ll}
(\lambda z.\overline{P})\ (\mathsf{I}\ (\lambda x.\overline{M})\ \overline{V}) & \longrightarrow^*_{\mathsf{assoc}\beta} (\lambda z.\overline{P})\ ((\lambda x.\overline{M})\ \overline{V}) \\
\left\{{}^{x}\!/\!_{y}\right\}\overline{N} & \longrightarrow^*_{\mathsf{assoc}\beta} \left\{{}^{x}\!/\!_{y}\right\}\overline{N} \\
\left\{{}^{\overline{M}}\!/\!_{y}\right\}y & \longrightarrow^*_{\mathsf{assoc}\beta} \overline{M}
\end{array}
$$

$$
\begin{array}{lll}
(\lambda x.\overline{N})\ ((\lambda y.\overline{P})\ (z\ \overline{V})) & \longrightarrow^*_{\mathsf{assoc}\beta} (\lambda y.(\lambda x.\overline{N})\ \overline{P})\ (z\ \overline{V}) & \\
(\lambda x.N)\ ((\lambda z.\overline{P})\ (\overline{V'}\ \overline{V})) & \longrightarrow^*_{\mathsf{assoc}\beta} (\lambda z.(\lambda x.\overline{N})\ \overline{P})\ (\overline{V'}\ \overline{V}) & \\
(\lambda x.\overline{N})\ (\left\{{}^{\overline{M}}\!/\!_{y}\right\}\overline{P}) & \longrightarrow^*_{\mathsf{assoc}\beta} \left\{{}^{\overline{M}}\!/\!_{y}\right\}((\lambda x.\overline{N})\ \overline{P}) & \text{if } \langle M \dagger y.P\rangle \text{ is pseudo principal} \\
 & & \text{(and thus } \langle M \dagger y.\langle P \dagger x.N\rangle\rangle \text{ is pseudo principal)} \\
(\lambda x.\overline{N})\ ((\lambda y.\overline{P})\ \overline{M}) & \longrightarrow^*_{\mathsf{assoc}\beta} (\lambda y.(\lambda x.\overline{N})\ \overline{P})\ \overline{M} & \text{if not} \\
\left\{{}^{\mathsf{I}\ \lambda y.\overline{M}}\!/\!_{x}\right\}\overline{N} & \longrightarrow^*_{\mathsf{assoc}\beta} \left\{{}^{\mathsf{I}\ \lambda y.\overline{M}}\!/\!_{x}\right\}\overline{N} &
\end{array}
$$

$$
\begin{array}{lll}
\left\{{}^{\overline{V}}\!/\!_{x}\right\}x & \longrightarrow^*_{\mathsf{assoc}\beta} \overline{V} & \\
\left\{{}^{\overline{V}}\!/\!_{x}\right\}y & \longrightarrow^*_{\mathsf{assoc}\beta} y & \\
\left\{{}^{\overline{V}}\!/\!_{x}\right\}(\mathsf{I}\ \lambda y.\overline{M}) & \longrightarrow^*_{\mathsf{assoc}\beta} \mathsf{I}\ \lambda y.\left\{{}^{\overline{V}}\!/\!_{x}\right\}\overline{M} & \\
\hline
\left\{{}^{\overline{V}}\!/\!_{x}\right\}\overline{V'} & \longrightarrow^*_{\mathsf{assoc}\beta} \left\{{}^{\overline{V}}\!/\!_{x}\right\}\overline{V'} & \\
\left\{{}^{\overline{V}}\!/\!_{x}\right\}((\lambda z.\overline{P})\ (x\ \overline{V'})) & \longrightarrow^*_{\mathsf{assoc}\beta} (\lambda z.\left\{{}^{\overline{V}}\!/\!_{x}\right\}\overline{P})\ (\overline{V}\ \left\{{}^{\overline{V}}\!/\!_{x}\right\}\overline{V'}) & \\
\left\{{}^{\overline{V}}\!/\!_{x}\right\}((\lambda z.\overline{P})\ (x'\ \overline{V'})) & \longrightarrow^*_{\mathsf{assoc}\beta} (\lambda z.\left\{{}^{\overline{V}}\!/\!_{x}\right\}\overline{P})\ (x'\ \left\{{}^{\overline{V}}\!/\!_{x}\right\}\overline{V'}) & \\
\left\{{}^{\overline{V}}\!/\!_{x}\right\}\left\{{}^{\overline{M}}\!/\!_{y}\right\}\overline{P} & \longrightarrow^*_{\mathsf{assoc}\beta} \left\{{}^{\left\{{}^{\overline{V}}\!/\!_{x}\right\}\overline{M}}\!/\!_{y}\right\}\left\{{}^{\overline{V}}\!/\!_{x}\right\}\overline{P} & \text{if } \langle M \dagger y.P\rangle \text{ is pseudo principal} \\
 & & \text{(and thus } \langle\langle V \nwarrow x.M\rangle \dagger y.\langle V \nwarrow x.P\rangle\rangle \text{ is pseudo principal)} \\
\left\{{}^{\overline{V}}\!/\!_{x}\right\}((\lambda y.\overline{P})\ \overline{M}) & \longrightarrow^*_{\mathsf{assoc}\beta} (\lambda y.\left\{{}^{\overline{V}}\!/\!_{x}\right\}\overline{P})\ \left\{{}^{\overline{V}}\!/\!_{x}\right\}\overline{M} & \text{if not}
\end{array}
$$

The rest of the induction is straightforward, since $\longrightarrow_{\mathsf{assoc}\beta}$ is context-closed and principal cuts are stable under reduction (Remark 30). □

4.2 A labelled LPO

Definition 18 We define a first-order syntax by giving the following infinite signature:

$$\{\star/0, \mathsf{i}/1, \mathsf{ii}/2, \mathsf{C}^M_i/2, \mathsf{C}^M_{ii}/2, \mathsf{C}^M_{iii}/2, \mathsf{C}^M_{iv}/2\}$$

with M ranging over the set of λ-terms in SN^β.

We give them the following (terminating[1]) precedence:

$$\star \prec \mathsf{i} \prec \mathsf{ii} \prec \mathsf{C}^N_i \prec \mathsf{C}^N_{ii} \prec \mathsf{C}^N_{iii} \prec \mathsf{C}^N_{iv} \prec \mathsf{C}^M_i$$

[1] because of Theorem 28

for all $M, N \in \mathsf{SN}^{\mathsf{assoc}\beta}$ such that $M(\longrightarrow_{\mathsf{assoc}\beta} \cup \sqsupseteq)^* N$.

We now consider the (terminating) *lexicographic path ordering* induced by this precedence over first-order terms (see e.g. [Ter03]).

We now want to encode β-bounded $\lambda\mathsf{LJQ}$-terms into this first-order syntax to show that $\lambda\mathsf{LJQ}$-reduction decreases first-order encodings w.r.t. the LPO. For that we need to make a case distinction to encode $\langle N \dagger x.M\rangle$, as either a C_i-construct or a C_{iii}-one. Those to be encoded as a C_i-construct are identified as the set $\mathcal{HP}$ defined in Fig. 6.

$$\frac{}{\langle [V'] \dagger x.x[V, y.M]\rangle \in \mathcal{HP}}\ x \notin \mathsf{FV}(V) \cup \mathsf{FV}(M)$$

$$\frac{}{\left\langle V \nwarrow x.M \right\rangle \in \mathcal{HP}}\ M \in \mathcal{S}_x \qquad \frac{M \in \mathcal{HP}}{\langle M \dagger x.N\rangle \in \mathcal{HP}}$$

Figure 6: Set of "small" cuts

Remark 35 Note that $\mathcal{V} \cap \mathcal{HP} = \emptyset$. Consequently, if $\langle M \dagger x.N\rangle \in \mathcal{HP}$ with $M \in \mathcal{V}$, then it is of the form $\langle [V] \dagger x.x[V', y.P]\rangle$ with $x \notin \mathsf{FV}(V) \cup \mathsf{FV}(P)$.

Remark 36 If $M \longrightarrow^* M'$, $N \longrightarrow^* N'$, $V \longrightarrow^* V'$ and $\langle M \dagger x.N\rangle$ (resp. $\left\langle V \nwarrow x.N \right\rangle$) is in $\mathcal{HP}$, then so is $\langle M' \dagger x.N'\rangle$ (resp. $\left\langle V' \nwarrow x.N' \right\rangle$).

Remark 37 If M is in $\mathcal{HP}$ then so is $\{z/_y\}M$.

Fig. 7 then gives the encoding into the first-order syntax.

$$\begin{array}{lll}
\underline{x} & := & \star \\
\underline{\lambda x.M} & := & \mathsf{i}(\underline{M}) \\
\underline{\left\langle V \nwarrow x.V' \right\rangle} & := & \mathsf{C}_{iv}^{\overline{\left\langle V \nwarrow x.V' \right\rangle}}(\underline{V}, \underline{V'}) \\
\hline
\underline{[V]} & := & \mathsf{i}(\underline{V}) \\
\underline{x[V, y.M]} & := & \mathsf{ii}(\underline{V}, \underline{M}) \\
\underline{\langle N \dagger x.M\rangle} & := & \mathsf{C}_{i}^{\overline{\langle N \dagger x.M\rangle}}(\underline{N}, \underline{M}) \qquad \langle N \dagger x.M\rangle \in \mathcal{HP} \\
\underline{\langle N \dagger x.M\rangle} & := & \mathsf{C}_{iii}^{\overline{\langle N \dagger x.M\rangle}}(\underline{N}, \underline{M}) \qquad \text{otherwise} \\
\underline{\left\langle V \nwarrow x.M \right\rangle} & := & \mathsf{C}_{ii}^{\overline{\left\langle V \nwarrow x.M \right\rangle}}(\underline{V}, \underline{M}) \qquad \left\langle V \nwarrow x.M \right\rangle \in \mathcal{HP} \\
\underline{\left\langle V \nwarrow x.M \right\rangle} & := & \mathsf{C}_{iv}^{\overline{\left\langle V \nwarrow x.M \right\rangle}}(\underline{V}, \underline{M}) \qquad \text{otherwise}
\end{array}$$

Figure 7: Encoding into the first-order syntax

Remark 38 By Remark 37, $\underline{\{x/_y\}M} = \underline{M}$.

Remark 39

1. $\mathsf{C}_{iii}^{(\lambda x.\overline{M})\ \overline{N}}(\underline{N}, \underline{M})\ (\gg \cup =)\ \underline{\langle N \dagger x.M\rangle}\ (\gg \cup =)\ \mathsf{C}_{i}^{\left\{\overline{N}/_x\right\}\overline{M}}(\underline{N}, \underline{M})$

2. $\mathsf{C}_{iv}^{\left\{\overline{N}/_x\right\}\overline{M}}(\underline{N}, \underline{M})\ (\gg \cup =)\ \underline{\left\langle N \nwarrow x.M \right\rangle}\ (\gg \cup =)\ \mathsf{C}_{ii}^{\left\{\overline{N}/_x\right\}\overline{M}}(\underline{N}, \underline{M})$

Theorem 40 *If $M \in \mathcal{B}$ (resp. $V \in \mathcal{B}$) and $M \longrightarrow_{\lambda\mathsf{LJQ}} N$ (resp. $V \longrightarrow_{\lambda\mathsf{LJQ}} V'$), then $N \in \mathcal{B}$ (resp. $V' \in \mathcal{B}$) and $\underline{M} \gg \underline{N}$ (resp. $\underline{V} \gg \underline{V'}$).*

Proof: By induction on the reduction steps $M \longrightarrow_{\lambda\mathsf{LJQ}} N$ (resp. $V \longrightarrow_{\lambda\mathsf{LJQ}} V'$). The following table gives the base case of the induction: for each reduction rule, the left-hand side shows a term that is (by Remarks 33 and 39) greater than or equal to $\underline{M}$ (resp. $\underline{V}$), while the right-hand side shows a term that is (by Remark 33 and 39) greater than or equal to $\underline{N}$ (resp. $\underline{V'}$).

$$\mathsf{C}_i^{(\lambda z.\overline{P})\ (\mathsf{I}\ (\lambda x.\overline{M})\ \overline{V})}(\mathsf{i}(\mathsf{i}(\underline{M})), \mathsf{ii}(\underline{V}, \underline{P})) \gg \mathsf{C}_{iii}^{(\lambda z.\overline{P})\ ((\lambda x.\overline{M})\ \overline{V})}(\mathsf{C}_{iii}^{(\lambda x.\overline{M})\ \overline{V}}(\mathsf{i}(\underline{V}), \underline{M}), \underline{P})$$

$$\mathsf{C}_{iv}^{\{{}^{x}\!/_{y}\}\overline{N}}(\star, \underline{N}) \gg \underline{N}$$

$$\mathsf{C}_{iv}^{\left\{{}^{\overline{M}}\!/_{y}\right\}y}(\underline{M}, \star) \gg \underline{M}$$

$$\mathsf{C}_{iii}^{(\lambda x.\overline{N})\ ((\lambda y.\overline{P})\ (z\ \overline{V}))}(\mathsf{ii}(\underline{V}, \underline{P}), \underline{N}) \gg \mathsf{ii}(\underline{V}, \mathsf{C}_{iii}^{(\lambda y.(\lambda x.\overline{N})\ \overline{P})\ (z\ \overline{V})}(\underline{P}, \underline{N}))$$

$$\mathsf{C}_i^{(\lambda x.\overline{N})\ L}(\mathsf{C}_i^L(\mathsf{i}(V'), \mathsf{ii}(\underline{V}, \underline{P})), \underline{N}) \gg \mathsf{C}_i^{(\lambda z.L')\ (\overline{V'}\ \overline{V})}(\mathsf{i}(\underline{V'}), \mathsf{C}_{iii}^{L'}(\underline{P}, \underline{N}))$$

with $L = (\lambda z.\overline{P})\ (\overline{V'}\ \overline{V})$ and $L' = (\lambda x.\overline{N})\ \overline{P}$

The next rule splits into two cases:[2]

$$\mathsf{C}_i^{(\lambda x.\overline{N})\ L}(\mathsf{C}_i^L(\underline{M}, \underline{P}), \underline{N}) \gg \mathsf{C}_i^{(\lambda y.L')\ \overline{M}}(\underline{M}, \mathsf{C}_{iii}^{L'}(\underline{P}, \underline{N})) \quad \text{if } \langle M \dagger y.P\rangle \in \mathcal{HP}$$

with $L = (\lambda y.\overline{P})\ \overline{M}$ and $L' = (\lambda x.\overline{N})\ \overline{P}$

$$\mathsf{C}_{iii}^{(\lambda x.\overline{N})\ L}(\mathsf{C}_{iii}^L(\underline{M}, \underline{P}), \underline{N}) \gg \mathsf{C}_{iii}^{L'}(\underline{M}, \mathsf{C}_{iii}^{(\lambda x.\overline{N})\ \overline{P}}(\underline{P}, \underline{N})) \quad \text{if not, with,}$$

if $\langle M \dagger y.P\rangle$ is pseudo principal, $L = \left\{{}^{\overline{M}}\!/_{y}\right\}\overline{P}$ and $L' = \left\{{}^{\overline{M}}\!/_{y}\right\}((\lambda x.\overline{N})\ \overline{P})$

otherwise $L = (\lambda y.\overline{P})\ \overline{M}$ and $L' = (\lambda y.(\lambda x.\overline{N})\ \overline{P})\ \overline{M}$

The last rule, again, splits into two cases:[3]

$$\mathsf{C}_{iii}^{\left\{{}^{\mathsf{I}\ \lambda y.\overline{M}}\!/_{x}\right\}\overline{N}}(\mathsf{i}(\mathsf{i}(\underline{M})), \underline{N}) \gg \mathsf{C}_{ii}^{\left\{{}^{\mathsf{I}\ \lambda y.\overline{M}}\!/_{x}\right\}\overline{N}}(\mathsf{i}(\underline{M}), \underline{N}) \quad \text{if } N \in \mathcal{S}_x$$

$$\mathsf{C}_{iii}^{(\lambda x.\overline{N})\ (\mathsf{I}\ \lambda y.\overline{M})}(\mathsf{i}(\mathsf{i}(\underline{M})), \underline{N}) \gg \mathsf{C}_{iv}^{\left\{{}^{\mathsf{I}\ \lambda y.\overline{M}}\!/_{x}\right\}\overline{N}}(\mathsf{i}(\underline{M}), \underline{N}) \quad \text{if not}$$

---

[2] Given the condition on $\langle M \dagger y.P\rangle$ for the rule to apply, $\langle M \dagger y.P\rangle \in \mathcal{HP}$ if and only if $M \in \mathcal{HP}$ if and only if $\langle M \dagger y.\langle P \dagger x.N\rangle\rangle \in \mathcal{HP}$ if and only if $\langle\langle M \dagger y.P\rangle \dagger x.N\rangle \in \mathcal{HP}$. Moreover, in the first case, $\langle M \dagger y.P\rangle \in \mathcal{HP}$ entails that it is not pseudo principal (by Remark 35), hence the shape of L (and note that $L \longrightarrow_{\mathsf{assoc}} (\lambda y.(\lambda x.\overline{N})\ \overline{P})\ \overline{M} \sqsupset L'$).

[3] In both cases of the last rule, we have encoded the redex as a C_{iii}-construct since it cannot be in $\mathcal{HP}$ (by Remark 35).

$$
\begin{array}{ll}
\mathsf{C}_{iv}^{\overline{V}}(\underline{V},\star) & \gg \underline{V} \\
\mathsf{C}_{iv}^{y}(\underline{V},\star) & \gg \star \\
\mathsf{C}_{iv}^{\left\{\overline{V}/_x\right\}(\mathsf{I}\ \lambda y.\overline{M})}(\underline{V},\mathsf{i}(\underline{M})) & \gg \mathsf{i}(\mathsf{C}_{iv}^{\left\{\overline{V}/_x\right\}\overline{M}}(\underline{V},\underline{M})) \\
\hline
\mathsf{C}_{iv}^{\left\{\overline{V}/_x\right\}\overline{V'}}(\underline{V},\mathsf{i}(\underline{V'})) & \gg \mathsf{i}(\mathsf{C}_{iv}^{\left\{\overline{V}/_x\right\}\overline{V'}}(\underline{V},\underline{V'})) \\
\mathsf{C}_{ii}^{L}(\underline{V},\mathsf{ii}(\underline{V'},\underline{P})) & \gg \mathsf{C}_{i}^{L}(\mathsf{i}(\underline{V}),\mathsf{ii}(\mathsf{C}_{iv}^{\left\{\overline{V}/_x\right\}\overline{V'}}(\underline{V},\underline{V'}),\mathsf{C}_{iv}^{\left\{\overline{V}/_x\right\}\overline{P}}(\underline{V},\underline{P}))) \\
 & \qquad \text{with } L = \left\{\overline{V}/_x\right\}((\lambda z.\overline{P})\ (x\ \overline{V'})) \\
\mathsf{C}_{iv}^{L}(\underline{V},\mathsf{ii}(\underline{V'},\underline{P})) & \gg \mathsf{ii}(\mathsf{C}_{iv}^{\left\{\overline{V}/_x\right\}\overline{V'}}(\underline{V},\underline{V'}),\mathsf{C}_{iv}^{\left\{\overline{V}/_x\right\}\overline{P}}(\underline{V},\underline{P})) \\
 & \qquad \text{with } L = \left\{\overline{V}/_x\right\}((\lambda z.\overline{P})\ (x'\ \overline{V'}))
\end{array}
$$

The last rule splits into two cases:[4]

$$
\begin{array}{ll}
\mathsf{C}_{ii}^{\left\{\overline{V}/_x\right\}L}(\underline{V},\mathsf{C}_{i}^{L}(\underline{M},\underline{P})) & \gg \mathsf{C}_{i}^{\left\{\overline{V}/_x\right\}L}(\mathsf{C}_{ii}^{L_M}(\underline{V},\underline{M}),\mathsf{C}_{iv}^{L_P}(\underline{V},\underline{P})) \quad \text{if } \langle M \dagger y.P\rangle \in \mathcal{S}_x \\
 & \text{with } L = (\lambda y.\overline{P})\ \overline{M},\ L_M = \left\{\overline{V}/_x\right\}\overline{M}, \text{ and } L_P = \left\{\overline{V}/_x\right\}\overline{P} \\
\mathsf{C}_{iv}^{\left\{\overline{V}/_x\right\}L}(\underline{V},\mathsf{C}_{i}^{L}(\underline{M},\underline{P})) & \gg \mathsf{C}_{iii}^{\left\{\overline{V}/_x\right\}L}(\mathsf{C}_{iv}^{L_M}(\underline{V},\underline{M}),\mathsf{C}_{iv}^{L_P}(\underline{V},\underline{P})) \quad \text{if not} \\
\multicolumn{2}{l}{\quad \text{with } L = (\lambda y.\overline{P})\ \overline{M} \text{ or } L = \left\{\overline{M}/_y\right\}\overline{P},\ L_M = \left\{\overline{V}/_x\right\}\overline{M}, \text{ and } L_P = \left\{\overline{V}/_x\right\}\overline{P}}
\end{array}
$$

The inductive argument is straightforward, given that $\mathcal{S}_x$ and $\mathcal{HP}$ are stable under reduction and the fact that, by Theorem 34, internal reductions can only decrease labels with respect to $(\longrightarrow_{\mathsf{assoc}\beta} \cup \sqsupset)$. □

Corollary 41 $\mathcal{B} \subseteq \mathsf{SN}$

Proof: By Theorem 8: the decreasing and terminating LPO simulates λLJQ-reductions through $\underline{(\ \cdot\ )}$. □

4.3 Conclusion

Lemma 42 *If* $\Gamma \vdash M : A$ *then, for every* $N \sqsubseteq M$ *(resp.* $V \sqsubseteq M$*),* $\Gamma \vdash_\lambda \overline{M} : A$ *(resp.* $\Gamma \vdash_\lambda \overline{V} : A$*) in the simply-typed* λ*-calculus.*

Proof: Straightforward induction on the typing tree, using the typing property of substitution in λ-calculus. □

Theorem 43 *If* $\Gamma \vdash M : A$ *then* $M \in \mathsf{SN}$*.*

Proof: By Lemma 42 and the strong normalisation of the simply-typed λ-calculus, $M \in \mathcal{B}$ so $M \in \mathsf{SN}$ by Corollary 41. □

References

[DL06] R. Dyckhoff and S. Lengrand. LJQ, a strongly focused calculus for intuitionistic logic. In A. Beckmann, U. Berger, B. Loewe, and J. V. Tucker, editors, *Proc. of the 2nd Conf. on Computability in Europe (CiE'06)*, volume 3988 of *LNCS*, pages 173–185. Springer-Verlag, July 2006.

[4] If $\langle M \dagger y.P\rangle \in \mathcal{S}_x$, first it cannot be pseudo principal (Remark 29), and second we also have $M \in \mathcal{S}_x$ so $\langle\langle V \nwarrow x.M\rangle \dagger y.\langle V \nwarrow x.P\rangle\rangle \in \mathcal{HP}$. Also note that $L_P \sqsubset \left\{\overline{V}/_x\right\} L$.

[DL07] R. Dyckhoff and S. Lengrand. Call-by-value λ-calculus and LJQ. *J. Logic Comput.*, 2007. Accepted for publication.

[Len06] S. Lengrand. *Normalisation & Equivalence in Proof Theory & Type Theory.* PhD thesis, Université Paris 7 & University of St Andrews, 2006.

[Mog88] E. Moggi. Computational lambda-calculus and monads. Report ECS-LFCS-88-66, University of Edinburgh, Edinburgh, Scotland, October 1988.

[Ter03] Terese. *Term Rewriting Systems*, volume 55 of *Cambridge Tracts in Theoret. Comput. Sci.* Cambridge University Press, 2003.

A Examples

Example 1 We give examples of unwanted reductions (leading to non-termination), which motivated the side-conditions of two rules of $\lambda\mathsf{LJQ}$.

Suppose that W is an abstraction and N is a y-covalue. The reductions $\langle\langle[W] \dagger y.N\rangle \dagger z.P\rangle \longrightarrow \langle[W] \dagger y.\langle N \dagger z.P\rangle\rangle$ and $\langle[W] \dagger y.N\rangle \longrightarrow \big\langle[W] \nwarrow y.N\big\rangle$ are not valid (as the side-conditions of the rules are not satisfied). Our first-order encoding translates them to

$$\mathsf{C}_i^{(\lambda y.\overline{P})\ \left\{\overline{W}\!/\!_y\right\}\overline{N}}(\mathsf{C}_i^{\left\{\overline{W}\!/\!_y\right\}\overline{N}}(\mathsf{i}(\underline{W}),\underline{N}),\underline{P}) \not\gg \mathsf{C}_{iii}^{\left\{\overline{W}\!/\!_y\right\}((\lambda z.\overline{P})\ \overline{N})}(\mathsf{i}(\underline{W}),\mathsf{C}_{iii}^{(\lambda z.\overline{P})\ \overline{N}}(\underline{N},\underline{P}))$$

$$\mathsf{C}_i^{\left\{\overline{W}\!/\!_y\right\}\overline{N}}(\mathsf{i}(\underline{W}),\underline{N}) \not\gg \mathsf{C}_{ii}^{\left\{\overline{W}\!/\!_y\right\}\overline{N}}(\underline{W},\underline{N})$$

Indeed, non-termination of these reductions can be seen as we can turn $\langle\ \dagger\ .\ \rangle$ into $\big\langle\ \nwarrow\ .\ \big\rangle$ and back:

$$\begin{array}{ccc}
 & \langle\langle[W] \dagger y.N\rangle \dagger z.P\rangle & \\
\swarrow & & \searrow \\
\langle[W] \dagger y.\langle N \dagger z.P\rangle\rangle & & \big\langle\big\langle W \nwarrow y.N\big\rangle \dagger z.P\big\rangle \\
\downarrow & & \downarrow \\
\big\langle W \nwarrow y.\langle N \dagger z.P\rangle\big\rangle & & \langle\langle[W] \dagger y.N'\rangle \dagger z.P\rangle \\
\downarrow & & \\
\big\langle\big\langle W \nwarrow y.N\big\rangle \dagger z.P'\big\rangle & & \\
\downarrow & & \\
\langle\langle[W] \dagger y.N'\rangle \dagger z.P'\rangle & &
\end{array}$$

where $P' = \big\langle W \nwarrow y.P\big\rangle$ (but note that $y \notin \mathsf{FV}(P)$) and N' is the y-covalue obtained by pushing W inside the y-covalue N.

Example 2 On the other hand, the following reductions *are* allowed (under which the image by $\overline{(\ \cdot\ )}$ is unchanged).

$$\begin{array}{c}
\langle V \nwarrow x.\langle\langle [W] \dagger y.N\rangle \dagger z.P\rangle\rangle \\
\downarrow_{+} \\
\langle\langle [W'] \dagger y.\langle V \nwarrow x.N\rangle\rangle \dagger z.P'\rangle \\
\swarrow \qquad\qquad\qquad\qquad\qquad\qquad \searrow
\end{array}$$

$$\begin{array}{cc}
\langle [W'] \dagger y.\langle\langle V \nwarrow x.N\rangle \dagger z.P'\rangle\rangle & \langle\langle W' \nwarrow y.\langle V \nwarrow x.N\rangle\rangle \dagger z.P'\rangle \\
\downarrow & \downarrow \\
\langle [W'] \dagger y.\langle N' \dagger z.P'\rangle\rangle & \langle\langle W' \nwarrow y.N'\rangle \dagger z.P'\rangle \\
\downarrow & \downarrow \\
\langle W' \nwarrow y.\langle N' \dagger z.P'\rangle\rangle & \langle\langle [W'] \dagger y.N''\rangle \dagger z.P'\rangle \\
\downarrow & \\
\langle\langle W' \nwarrow y.N'\rangle \dagger z.P''\rangle & \\
\downarrow & \\
\langle\langle [W'] \dagger y.N''\rangle \dagger z.P''\rangle &
\end{array}$$

where $W' = \langle V \nwarrow x.W\rangle$, $P' = \langle V \nwarrow x.P\rangle$, $P'' = \langle W' \nwarrow y.P'\rangle$ (but note that $y \notin \mathsf{FV}(P')$), N' is the y-covalue obtained by pushing V inside the y-covalue N, and N'' is the y-covalue obtained by pushing W' inside the y-covalue N'.

This motivated the case distinctions leading to $\mathsf{C}_i, \mathsf{C}_{ii}, \mathsf{C}_{iii}$ and C_{iv} and their precedence. Indeed we have

$$\begin{array}{rl}
& \mathsf{C}_{iv}^{\{\overline{V}/x\}((\lambda z.\overline{P})\ \{\overline{W}/y\}\overline{N})}(\underline{V}, \mathsf{C}_{i}^{(\lambda y.\overline{P})\ \{\overline{W}/y\}\overline{N}}(\mathsf{C}_{i}^{\{\overline{W}/y\}\overline{N}}(\mathsf{i}(\underline{W}), \underline{N}), \underline{P})) \\
\gg & \mathsf{C}_{iii}^{(\lambda z.\overline{P'})\ \{\overline{W'}/y\}\{\overline{V}/x\}\overline{N}}(\mathsf{C}_{iii}^{\{\overline{W'}/y\}\{\overline{V}/x\}\overline{N}}(\mathsf{i}(\underline{W'}), \mathsf{C}_{iv}^{\{\overline{V}/x\}N}(\underline{V}, \underline{N})), \underline{P'})
\end{array}$$

In the first branch we then get

$$\begin{array}{rl}
\gg & \mathsf{C}_{iii}^{\{\overline{W'}/y\}((\lambda z.\overline{P'})\ \{\overline{V}/x\}\overline{N})}(\mathsf{i}(\underline{W'}), \mathsf{C}_{iii}^{(\lambda z.\overline{P'})\ \{\overline{V}/x\}\overline{N}}(\mathsf{C}_{iv}^{\{\overline{V}/x\}N}(\underline{V}, \underline{N}), \underline{P'})) \\
\gg & \mathsf{C}_{iii}^{\{\overline{W'}/y\}((\lambda z.\overline{P'})\ \overline{N'})}(\mathsf{i}(\underline{W'}), \mathsf{C}_{iii}^{(\lambda z.\overline{P'})\ \overline{N'}}(\underline{N'}, \underline{P'})) \\
\gg & \mathsf{C}_{ii}^{\{\overline{W'}/y\}((\lambda z.\overline{P'})\ \overline{N'})}(\underline{W'}, \mathsf{C}_{iii}^{(\lambda z.\overline{P'})\ \overline{N'}}(\underline{N'}, \underline{P'})) \\
\gg & \mathsf{C}_{i}^{(\lambda z.\overline{P''})\ \{\overline{W'}/y\}\overline{N'}}(\mathsf{C}_{ii}^{\{\overline{W'}/y\}\overline{N'}}(\underline{W'}, \underline{N'}), \underline{P''}) \\
\gg & \mathsf{C}_{i}^{(\lambda z.\overline{P''})\ \{\overline{W'}/y\}\overline{N''}}(\mathsf{C}_{i}^{\{\overline{W'}/y\}\overline{N''}}(\mathsf{i}(\underline{W'}), \underline{N''}), \underline{P''})
\end{array}$$

In the second branch we then get

$$\begin{array}{rl}
\gg & \mathsf{C}_{iii}^{(\lambda z.\overline{P'})\ \{\overline{W'}/y\}\{\overline{V}/x\}\overline{N}}(\mathsf{C}_{ii}^{\{\overline{W'}/y\}\{\overline{V}/x\}\overline{N}}(\underline{W'}, \mathsf{C}_{iv}^{\{\overline{V}/x\}N}(\underline{V}, \underline{N})), \underline{P'}) \\
\gg & \mathsf{C}_{iii}^{(\lambda z.\overline{P'})\ \{\overline{W'}/y\}\overline{N'}}(\mathsf{C}_{ii}^{\{\overline{W'}/y\}\overline{N'}}(\underline{W'}, \underline{N'}), \underline{P'}) \\
\gg & \mathsf{C}_{iii}^{(\lambda z.\overline{P'})\ \{\overline{W'}/y\}\overline{N''}}(\mathsf{C}_{i}^{\{\overline{W'}/y\}\overline{N''}}(\mathsf{i}(\underline{W'}), \underline{N'}), \underline{P'})
\end{array}$$

In this example, we performed reductions dangerously close to the forbidden ones above (destroying or activating the principal cut), and both branches look like loops that turn $\langle\ \dagger\ .\ \rangle$ into $\langle\ \nwarrow\ .\ \rangle$ and back, except that this time, we have consumed V along the way.

Indeed, the first-order terms we have produced along the reductions are rather big (note the C_{iii}-constructs) in comparison to $\mathsf{C}_{i}^{(\lambda y.\overline{P})\ \{\overline{W}/y\}\overline{N}}(\mathsf{C}_{i}^{\{\overline{W}/y\}\overline{N}}(\mathsf{i}(\underline{W}), \underline{N}), \underline{P})$, but in fact they are never compared against it but against the C_{iv}-construct of $\langle V \nwarrow x.\rangle$. In other words, we have been able to perform this dangerous reductions only at the cost of pushing V in, and such things to push are not in infinite supplies.