Intuitionistic proof-search in SMT-solving / the calculus that was discovered (at least) 6 times

in memoriam Roy Dyckhoff

Stéphane Graham-Lengrand

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A calculus rediscovered many times

(Sequent) Calculus for Intuitionistic Propositional Logic

- Vorob'ev in the 50s
- Hudelmaier (88)
- Dyckhoff (90)
- Paulson (91)
- Lincoln-Scedrov-Shankar (91) (with a linear logic approach)

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Called

- LJT by Hudelmaier and then Dyckhoff (T for "Terminating", nothing to do with LJT from the linear logic tradition),
- G4ip by Troelstra-Schwichtenberg.
- "Contraction-free sequent calculus"
- "(Hudelmaier's) Depth-bounded sequent calculus"

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Each time, the calculus comes up with slight variations

Intuitionistic Sequent Calculus

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}$$

$$\frac{}{\bot, \Gamma \vdash C} \qquad \frac{(A \wedge B), A, B, \Gamma \vdash C}{(A \wedge B), \Gamma \vdash C} \qquad \frac{(A \vee B), A, \Gamma \vdash C \quad (A \vee B), B, \Gamma \vdash C}{(A \vee B), \Gamma \vdash C}$$

$$\frac{(C' \Rightarrow C), \Gamma \vdash C' \quad (C' \Rightarrow C), C, \Gamma \vdash D}{(C' \Rightarrow C), \Gamma \vdash D}$$

$$\frac{}{a, \Gamma \vdash a}$$

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Variant: atom a can be generalised as formula A (still sound)

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Variant: atom a can be generalised as formula A (still sound)
"*Contraction*": left-introduced formula is contracted with the occurrence already appearing in the premisses (different from linear logic, where some "context-sharing rules" hide contractions)

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Remark: correspondences with Natural Deduction tend to favour the presence of a main formula's duplicate in the premisses.

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Are the duplicates/contractions necessary for completeness?

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G4ip

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$$\frac{?}{(C' \Rightarrow C), \Gamma \vdash D}$$

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$$\frac{C, a, \Gamma \vdash D}{(a \Rightarrow C), a, \Gamma \vdash D}$$

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G4ip

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- Obviously sound

G4ip

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- Obviously sound
- Complete?

G4ip

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- Obviously sound
- Complete? rule permutation argument (Dyckhoff'92,'17), cut-elimination (Dyckhoff-Negri'00, Dyckhoff-SGL-Kesner'06)
- Connection with focused seq. calculus LJQ (Dyckhoff-SGL'06)

Good properties

- In each rule, each premiss is "smaller" than the conclusion (for the multiset order on the formulae present in the sequent) $\Rightarrow$ The height (aka depth) of proof-trees (for sequent $\Gamma \vdash A$) is bounded: it is a "depth-bounded sequent calculus".
$\Rightarrow$ "Root-first proof-search" (Roy's preferred terminology) terminates and constitutes decision procedure for provability of IPL (only finitely many trees of height $\leq$ bound)
- Each rule is invertible (if the conclusion is provable then so are the premisses), except (the $\vee$-right rules and)

$$\frac{A, (B \Rightarrow C), \Gamma \vdash B \quad C, \Gamma \vdash D}{((A \Rightarrow B) \Rightarrow C), \Gamma \vdash D}$$

which is *semi-invertible*: if the conclusion is provable, then so is the right premiss (the left premiss can be considered the side-condition of an invertible 1-premiss rule)

A generalised version of the semi-invertible rule

$$\frac{A, (B \Rightarrow C), \Gamma \vdash B \qquad C \ , \Gamma \vdash D}{((A \Rightarrow B) \Rightarrow C), \Gamma \vdash D}$$

A generalised version of the semi-invertible rule

$$\frac{a_1, \ldots, a_n,\ A, (B \Rightarrow C), \Gamma \vdash B \qquad\qquad (a_1 \wedge \cdots \wedge a_n \Rightarrow C), \Gamma \vdash D}{((A \Rightarrow B) \Rightarrow C), \Gamma \vdash D}$$

Still sound.

A generalised version of the semi-invertible rule

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Still sound. Derivable with a cut:

$$\dfrac{\dfrac{\dfrac{a_1, \ldots, a_n, A, (B \Rightarrow C), \Gamma \vdash B \qquad \overline{C, a_1, \ldots, a_n, \Gamma \vdash C}}{a_1, \ldots, a_n, ((A \Rightarrow B) \Rightarrow C), \Gamma \vdash C}}{((A \Rightarrow B) \Rightarrow C), \Gamma \vdash a_1 \wedge \cdots \wedge a_n \Rightarrow C} \qquad (a_1 \wedge \cdots \wedge a_n \Rightarrow C), \Gamma \vdash D}{((A \Rightarrow B) \Rightarrow C), \Gamma \vdash D}$$

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Remarks:

- If $n \neq 0$, rule is not necessarily semi-invertible.

A generalised version of the semi-invertible rule

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- If $n \neq 0$, rule is not necessarily semi-invertible. Fixed.
- More problematic: with or without the fix, weight of premisses not necessarily smaller than weight of conclusion. Depth-boundedness is probably lost.

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Remarks:

- If $n \neq 0$, rule is not necessarily semi-invertible. Fixed.
- More problematic: with or without the fix, weight of premisses not necessarily smaller than weight of conclusion. Depth-boundedness is probably lost.
- Is it not a bad idea to use the cut-rule in proof-search? How do we come up with $a_1, \ldots, a_n$?

A generalised version of the semi-invertible rule

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In conclusion: this generalisation sounds like a terrible idea.

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Recovering termination:

- Let's impose (1) that $(a_1 \wedge \cdots \wedge a_n \Rightarrow C) \notin \Gamma$, otherwise the right premiss is identical/equivalent to the conclusion.

A generalised version of the semi-invertible rule

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Recovering termination:

- Let's impose (1) that $(a_1 \wedge \cdots \wedge a_n \Rightarrow C) \notin \Gamma$, otherwise the right premiss is identical/equivalent to the conclusion.
- Using cuts in root-first proof-search is cumbersome unless we have a magic trick to produce the cut-formula (here: the $a_1, \ldots, a_n$)

A generalised version of the semi-invertible rule

$$\frac{a_1,\ldots,a_n,A,(B\Rightarrow C),\Gamma\vdash B \qquad ((A\Rightarrow B)\Rightarrow C),(a_1\wedge\cdots\wedge a_n\Rightarrow C),\Gamma\vdash D}{((A\Rightarrow B)\Rightarrow C),\Gamma\vdash D}$$

Recovering termination:

- Let's impose (1) that $(a_1\wedge\cdots\wedge a_n\Rightarrow C)\notin\Gamma$, otherwise the right premiss is identical/equivalent to the conclusion.
- Using cuts in root-first proof-search is cumbersome unless we have a magic trick to produce the cut-formula (here: the $a_1,\ldots,a_n$)
- Let's impose (2) that $a_1,\ldots,a_n$ are atoms present in the conclusion.

A generalised version of the semi-invertible rule

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Recovering termination:

- Let's impose (1) that $(a_1\wedge\cdots\wedge a_n{\Rightarrow}C) \notin \Gamma$, otherwise the right premiss is identical/equivalent to the conclusion.
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(1) and (2) recover termination (& preserve completeness).
Still, a lot of choices for $\{a_1, \ldots, a_n\}$

Restricting $\{a_1, \ldots, a_n\}$

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Then what?

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Conclusion: they are very good intuitionistic provers
... but are limited to proving sequents of that form.

Preprocessing

It's the preprocess that implements "every formula F can be transformed into an equisatisfiable* CNF $C_1 \wedge \cdots \wedge C_n$" that uses classical reasoning.

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In the intuitionistic case, every formula F can be transformed into an (intuitionistically) equiprovable sequent $\Gamma_{\text{imp}}, \Gamma_{\text{flat}} \vdash d$ with

- d an atom
- Γ_{flat} made of *flat clauses*: $(a_1 \wedge \cdots \wedge a_n) \Rightarrow (b_1 \vee \cdots \vee b_m)$
- Γ_{imp} made of *implication clauses*: $((a \Rightarrow b) \Rightarrow c)$

Idea for proof-search:

- flat clauses are treated eagerly, to see if, by chance, $\Gamma_{\text{flat}} \vdash d$ is provable, using e.g., a SAT-solver.
- implication clauses treated lazily, using the (generalised) G4ip rule.

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Note 3: some of these rules were already presented by Vorob'ev in the form of pre-processing

From G4ip to SMT solving

With pre-processing the rule becomes

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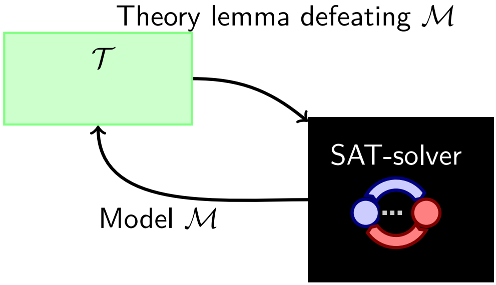
- thinking in terms of root-first proof-search, implemented recursively, the right premiss really corresponds to a tail call
  (i.e., a while loop)

From G4ip to SMT solving

The spine shape describes a traditional SMT algorithm: DPLL($\mathcal{T}$).



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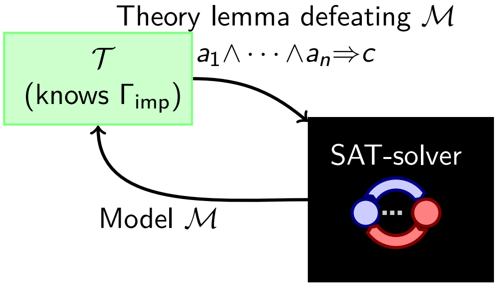

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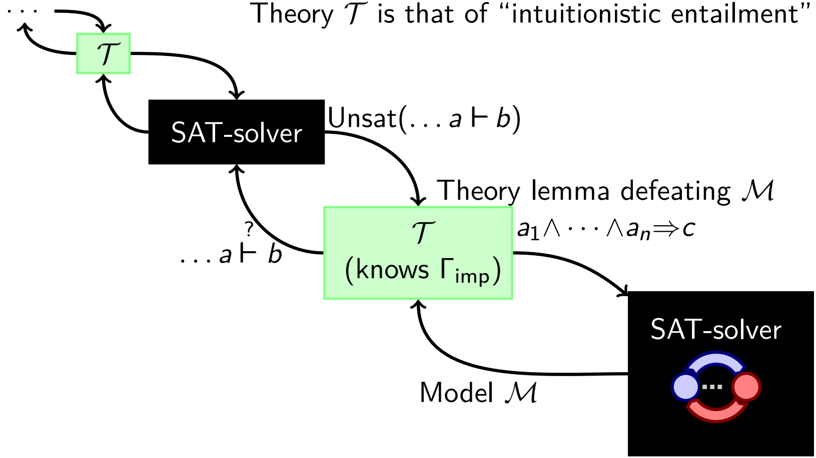

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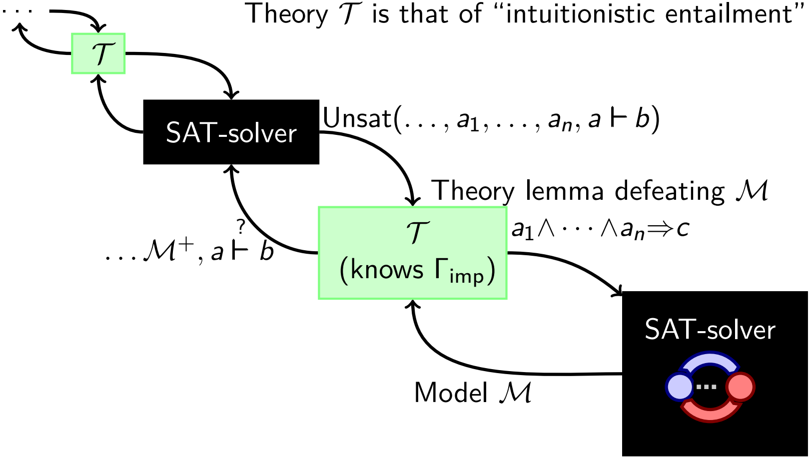

...except the "theory reasoning" that understands Γ_{imp} recursively relies on general provability.

And finally! we have the magic trick to pick $\{a_1, \ldots, a_n\}$: those atoms interpreted as true in $\mathcal{M}$ that were useful to prove $\ldots a \vdash b$

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From G4ip to SMT solving

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From G4ip to SMT solving

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   Then go back to the SAT-solver (1.)

Note: by construction, the learnt clause could not already be in Γ_{flat} otherwise the SAT solver would not have proposed model $\mathcal{M}$

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Why 1. ?
A SAT-solver has an internal learning mechanism.
It would be good if whatever is learnt by the SAT-solver of the recursive call could be shared with the SAT-solver of the caller.

Recursivity

Thinking in terms of root-first proof-search, implemented recursively, the right premiss really corresponds to a tail call (a while loop / the while loop of the SAT-solver):

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So the question of recursivity is really about the left premiss.
So what really changes between the SAT-solver of the caller and that of the callee? Mostly:

- the addition of a
- the addition of $\neg b$
- most most most importantly: the removal of $\neg d$

Recursive SAT-solver is not allowed to exploit $\neg d$ to get UNSAT

Recursivity

Actually:

- Claessen and Dosén actually reuse the same SAT-solver for the recursive call, popping $\neg d$, pushing $a, \neg b$, so that what is learnt from each run (by the standard learning mechanisms of SAT-solving) is shared between the different runs.

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An "intuitionistic DPLL" would have to integrate some mechanism equivalent to making $\neg d$ unusable in (the part of the computation corresponding to) the recursive call.

Model-constructing SMT

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We use an approach to SMT-solving recently developed, model-constructing satisfiability (MCSAT), in which theory reasoning is inside the DPLL loop.

Just like DPLL tries to assign Boolean values to Boolean variables in order to build a (counter-)model of the sequent to be proved, MCSAT also tries to assign theory values to theory variables, e.g., $x \mapsto \frac{3}{4}$ for a rational variable x (if the theory were for instance arithmetic)

For our purpose, we need to find the "right" notion of value for our intuitionistic formulae, so that

- Proving $\ldots \vdash d$ forces an assignment "similar to" $d \mapsto 0$
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One way to do that is to tag the Boolean values with a label that keeps track of how we went up into left branches, e.g., assign $d \mapsto 0^{w}$ and then assign $a \mapsto 1^{w(a \Rightarrow b)}$ and $b \mapsto 0^{w(a \Rightarrow b)}$ where the label w for d's assignment is changed into a label $w(a \Rightarrow b)$ to mark the new jump into the left premiss of the G4 rule for $(a \Rightarrow b) \Rightarrow c$.

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Then one realises that the labels are the worlds from a Kripke model being built: Going into the left premiss creates a label extension $w(a \Rightarrow b)$ that says "pick a world above w where a is true but b is false, then try to find a contradiction from there".

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Ongoing work.
We also try to bypass the preprocessing and directly work on the input formulae.

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