A Mechanically Assisted Examination of Begging the Question in Anselm's Ontological Argument

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Abstract

I use mechanized verification to examine several first- and higher-order formalizations of Anselm's Ontological Argument against the charge of begging the question. I propose three different criteria for a premise to beg the question in fully formal proofs and find that one or another applies to all the formalizations examined. My purpose is to demonstrate that mechanized verification provides an effective and reliable technique to perform these analyses; readers may decide whether the forms of question begging so identified affect their interest in the Argument or its various formalizations.

1 Introduction

I assume readers have some familiarity with St. Anselm's 11'th Century Ontological Argument for the existence of God [2]; a simplified translation from the original Latin of Anselm's *Proslogion* is given in Figure 1, with some alternative readings in square parentheses. This version of the argument appears in Chapter II of the Proslogion; another version appears in Chapter III and speaks of the *necessary* existence of God. Many authors have examined the Argument, in both its forms; in recent years, most begin by rendering it in modern logic, employing varying degrees of formality. The Proslogion II argument is traditionally rendered in first-order logic while propositional modal logic is used for that of Proslogion III. More recently,

I am grateful to Richard Campbell of the Australian National University for stimulating discussion on these topics, to my colleagues Sam Owre and N. Shankar for many useful conversations on PVS and logic, and to the anonymous reviewers for very helpful comments.

We can conceive of [something/that] than which there is no greater

If that thing does not exist in reality, then we can conceive of a greater thing—namely, something [just like it] that does exist in reality

Thus, either the greatest thing exists in reality or it is not the greatest thing

Therefore the greatest thing exists in reality

[That's God]

Figure 1: The Ontological Argument

higher-order logic and quantified modal logic have been applied to the argument of Proslogion II. My focus here is the Proslogion II argument, represented completely formally in first- or higher-order logic, and explored with the aid of a mechanized verification system. Elsewhere [28], I use a verification system to examine renditions of the argument in modal logic, and also the argument of Proslogion III [29].

Verification systems are tools from computer science that are generally used for exploration and verification of software or hardware designs and algorithms; they comprise a specification language, which is essentially a rich (usually higher-order) logic, and a collection of powerful deductive engines (e.g., satisfiability solvers for combinations of theories, model checkers, and automated and interactive theorem provers). I have previously explored renditions of the Argument due to Oppenheimer and Zalta [20] and Eder and Ramharter [12] using the PVS verification system [25, 26], and those provide the basis for the work reported here. Benzmüller and Woltzenlogel-Paleo have likewise explored modal arguments due to Gödel and Scott using the Isabelle and Coq verification systems [5, 6].

Mechanized analysis confirms the conclusions of most earlier commentators: the Argument is valid. Attention therefore focuses on the premises and their interpretation. The premises are *a priori* (i.e., armchair speculation) and thus not suitable for empirical confirmation or refutation: it is up to the individual reader to accept or deny them. We may note, however, that the premises are consistent (i.e., they have a model), and this is among the topics that I previously subjected to mechanized examination [25] (as a byproduct, this examination demonstrates that the Argument does not compel a theological interpretation: in the exhibited model, that "than which there is no greater" is the number zero).

The Argument has been a topic of enduring fascination for nearly a thousand years; this is surely due to its derivation of a bold conclusion from unexceptionable premises, which naturally engenders a sense of disquiet: "The Argument does not, to a modern mind, seem very convincing, but it is easier to feel that it must be fallacious than it is to find out precisely where the fallacy lies" [31, page 472]. Many commentators have sought to identify a fallacy in the Argument or its interpretation (e.g., Kant famously denied it on the basis that "existence is not a predicate"). One direction of attack is to claim that the Argument "begs the question"[1]; that is, it essentially assumes what it sets out to prove [24, 33]. This is the charge that I examine here.

Begging the question has traditionally been discussed in the context of informal or semi-formal argumentation and dialectics [3, 4, 32, 34–36], where the concern is whether arguments that beg the question should be considered fallacious, or valid but unpersuasive, or may even be persuasive. Here, we examine question begging in the context of fully formal, mechanically checked proofs. My purpose is to provide techniques that can identify potential question begging in a systematic and fairly unequivocal manner. I do not condemn the forms of question begging that are identified; rather, my goal is to highlight them so that readers can make up their own minds and can also use these techniques to find other cases.

The paper is structured as follows. In the next section, I introduce a strict definition of "begging the question" and show that a rendition of the Argument due to Oppenheimer and Zalta [20] is vulnerable to this charge. Oppenheimer and Zalta use a definite description (i.e., they speak of "*that* than which there is no greater") and require an additional assumption to ensure this is well-defined. Eder and Ramharter argue that Anselm did not intend this interpretation (i.e., requires only "*something* than which there is no greater") [12, Section 2.3] and therefore dispense with the additional assumption of Oppenheimer and Zalta. In Section 3, I show that this version of the argument does not beg the question under the strict definition, but that it does so under a plausible weakening. In Section 4, I consider an alternative premise due to Eder and Ramharter and show that this does not beg the question under either of the previous interpretations, but I argue that it is at least as questionable as the premise that it replaces because it so perfectly discharges the main step of the proof that it seems reverse-engineered. I suggest a third interpretation for "begging the question" that matches this case. In Section 5, I consider the higher-order treatment of Eder and Ramharter [12, Section 3.3] and a variant derived from Campbell [7]; these proofs are more complicated than the first-order treatments but I show how the third interpretation for "begging the question" applies to them. I compare these interpretations to existing, mainly

[1]This phrase is widely misunderstood to mean "to invite the question." Its use in logic derives from medieval translations of Aristotle, where the Latin form *Petitio Principii* is also employed.

informal, accounts of what it means to "beg the question" in Section 6. Finally, in Section 7, I discuss some limitations and possible implications of this work.

2 Begging the Question: Strict Case

"Begging the question" is a form of circular reasoning in which we assume what we wish to prove. It is generally discussed in the context of informal argumentation where the premises and conclusion are expressed in natural language. In such cases, the question-begging premise may state the same idea as the conclusion, but in different terms, or it may contain superfluous or even false information, and there is much literature on how to diagnose and interpret such cases [3, 4, 32, 34–36]. That is not my focus. I am interested in formal, deductive arguments, and in criteria for begging the question that are themselves formal. Now, deductive proofs do not generate new knowledge—the conclusion is always implicit in the premises—but they can generate surprise or insight; I propose that criteria for question begging should focus on the extent to which either the conclusion or its proof are "so directly" represented in the premises as to vitiate the hope of surprise or insight.

The basic criterion for begging the question is that one of, or a collection of, the premises is equivalent to the conclusion. But if some of the premises are equivalent to the conclusion, what are the other premises for? Certainly we must need all the premises to deduce the conclusion (else we can eliminate some of them); thus we surely need all the other premises before we can establish that some of them are equivalent to the conclusion. Hence, the criteria for begging the question should apply *after* we have accepted the other premises. Thus, if C is our conclusion, Q our "questionable" premise (which may be a conjunction of simpler premises) and P our other premises, then Q begs the question if C is equivalent to Q, assuming P: i.e., $P \vdash C = Q$. Of course, this means we can prove C using Q: $P, Q \vdash C$, and we can also do the reverse: $P, C \vdash Q$.

Figure 2 presents Oppenheimer and Zalta's treatment of the Ontological Argument [20] in PVS. I will not describe this in detail, since it is explained at tutorial level [25] and used [26] elsewhere, but remark that the identifiers and constructs used here are from [26] rather than [25]. Briefly, the specification language of PVS is a strongly typed higher-order logic with predicate subtypes. This example uses only first order but does make essential use of predicate subtypes and the proof obligations that they can incur [30]. The uninterpreted type `beings` is used for those things that are "in the understanding" (i.e., "understandable beings"). Note that a question mark at the end of an identifier is merely a convention to indicate predicates (which in PVS are simply functions with return type `bool`). The predicate `God?`

```
oandz: THEORY
BEGIN

  beings: TYPE
  x, y: VAR beings

  >: (trichotomous?[beings])

  God?(x): bool = NOT EXISTS y: y > x

  re?(x): bool

  ExUnd: AXIOM EXISTS x: God?(x)

  Greater1: AXIOM FORALL x: (NOT re?(x) => EXISTS y: y > x)

  God_re: THEOREM re?(the(God?))

%---------------- Question Begging Analysis ----------------------

  Greater1_circ: THEOREM God_re => Greater1

END oandz
```

Figure 2: Oppenheimer and Zalta's Treatment, in PVS

recognizes those beings "than which there is no greater"; the axiom `ExUnd` asserts the existence of at least one such being; `the(God?)` is a definite description that identifies this being. PVS generates a proof obligation (not shown here) to ensure this being is unique (this is required by the predicate subtype used in the definition of `the`, which is part of the "Prelude" of standard theories built in to PVS) and `ExUnd` and the trichotomy of `>` (also from the Prelude)[2] are used to discharge this obligation. The uninterpreted predicate `re?` identifies those beings that exist "in reality" and the axiom `Greater1` asserts that if a being does not exist in reality, then there is a greater being.

The theorem `God_re` asserts that the being identified by the definite description `the(God?)` exists in reality. The PVS proof of this theorem is accomplished by the following commands.

PVS Proof

```
  (typepred "the(God?)")  (use "Greater1") (grind)
```

[2]Trichotomy is the condition `FORALL x, y:  x > y OR y > x OR x = y`.

These commands invoke the type associated with `the(God?)` (namely that it satisfies the predicate `God?`), the premise `Greater1`, and then apply the standard automated proof strategy of PVS, called `grind`. Almost all the proofs mentioned subsequently are similarly straightforward and we do not reproduce them in detail.

As first noted by Garbacz [14], the premise `Greater1` begs the question under the other assumptions of the formalization. We state the key implication as `Greater1_circ` (PVS Version 7 allows formula names to be used in expressions as shorthands for the formulas themselves) and prove it as follows.

PVS proof
```
(expand "God_re")
(expand "Greater1")
(typepred "the(God?)")
(grind :polarity? t)
(inst 1 "x!1")
(typepred ">")
(grind)
```

The first two steps expand the formula names to the formulas they represent, the `typepred` steps introduce the predicate subtypes associated with their arguments (namely, that `the(God?)` satisfies `God?` and that `>` is trichotomous) and the other steps perform quantifier reasoning and routine deductions.

Given that we have proved `God_re` from `Greater1` and vice-versa, we can easily prove they are equivalent. Thus, in the definition of "begging the question" given earlier, C here is `God_re`, Q is `Greater1` and P is the rest of the formalization (i.e., `ExUnd`, the definition of `God?`, and the predicate subtype `trichotomous?` asserted for `>`).

Update and Addendum to the published paper. Oppenheimer and Zalta recently attempted to rebut my claim that `Greater1` is question begging [21], and I regret that I ended this section without connecting the formal notion of strict begging, as established for `Greater1`, to an intuitive explanation why this premise should be considered question begging.

To see this, observe that `Greater1` could be replaced by its contrapositive.

```
Greater1cp: LEMMA FORALL x: (NOT EXISTS y: y > x) => re?(x)
```

And observe further that the left side of the implication is just `God?(x)` and so the formula can be rewritten as follows.

```
Greater1cp_alt: LEMMA FORALL x: God?(x) => re?(x)
```

My informal criterion for question begging, stated at the start of this section, concerns "the extent to which either the conclusion or its proof are 'so directly' represented in the premises as to vitiate the hope of surprise or insight." All will surely agree that the conclusion `God_re` is "so directly" represented in the premise `Greater1cp_alt` that there can be no surprise or insight, and so this premise begs the question. But `Greater1` is just an obfuscated version of `Greater1cp_alt` so it, too, surely begs the question.

Eder and Ramharter [12, Section 1.2(5)] observe that the conclusion to a deductively valid argument must be implicit or "contained" in the premises (otherwise, the reasoning would not be deductive), but an argument can only be persuasive or interesting "if it is possible to accept the premises without already recognizing that the conclusion follows from them. Thus, the desired conclusion has to be 'hidden' in the premises." In a footnote, they aver "Sometimes, proofs of the existence of God are accused of being question-begging, but this critique is untenable. It is odd to ask for a deductive argument whose conclusion is not contained in the premises. Logic cannot pull a rabbit out of the hat."

Eder and Ramharter are, of course, correct that the conclusion must be "contained" in the premises, but they are also correct that it should be "hidden," and so I dispute their claim that accusations of question begging are untenable for the Ontological Argument. I suggest that tests for question begging should expose the "hiding place" of the conclusion among the premises: if this is revealed as inadequate or contrived, then our interest in the argument, and its persuasiveness, are diminished. Here, the formal criterion of strict begging has identified that the conclusion is "hiding" in `Greater1`, and `Greater1cp_alt` reveals that its cover is rather flimsy.

When we have an argument with just two premises, that is $P_1, P_2 \vdash C$, it is necessary that P_2 entails $P_1 \supset C$, although P_2 may be expressed in a form that obfuscates this relationship. Hence it is inevitable that `Greater1` entails and obfuscates something like `Greater1cp_alt` and I suggest the degree to which it should be considered question begging is related to the degree to which the obfuscation seems flimsy once revealed or, perhaps equivalently, the entailment is trivial.
End of this addendum; some others appear later.

3 Begging the Question: Weaker Case

Eder and Ramharter [12, Section 2.3] claim that Anselm's Proslogion does not employ a definite description and that a correct reading is "*something* than which there is no greater." A suitable modification to the previous PVS theory is shown in Figure 3; the differences are that `>` is now an unconstrained relation on `beings`, and the conclusion is restated as the theorem `God_re_alt`. As before, this theorem is easily

```
eandr1: THEORY
BEGIN

  beings: TYPE
  x, y: VAR beings

  >(x, y): bool

  God?(x): bool = NOT EXISTS y: y > x

  re?(x): bool

  ExUnd: AXIOM EXISTS x: God?(x)

  Greater1: AXIOM FORALL x: (NOT re?(x) => EXISTS y: y > x)

  God_re_alt: THEOREM EXISTS x: God?(x) AND re?(x)

%---------------- Question Begging Analysis ----------------------

  Greater1_circ_alt1: THEOREM trichotomous?(>)
     IMPLIES God_re_alt => Greater1

  Greater1_circ_alt2: THEOREM (FORALL x, y: God?(x) => x>y or x=y)
     IMPLIES God_re_alt => Greater1

END eandr1

eandr1interp: THEORY
BEGIN

IMPORTING eandr1{{
  beings := nat,
  > := LAMBDA (x, y: nat): FALSE,
  re? := LAMBDA (x: nat): TRUE
}} AS model

END eandr1interp
```

Figure 3: Eder and Ramharter's First Order Treatment, in PVS

proved from the premises `ExUnd` and `Greater1` and the definition of `God?`. However, `Greater1` no longer strictly begs the question because it cannot be proved from the conclusion `God_re_alt`.

We can observe, however, that this specification of the Argument is very austere and imposes no constraints on the relation `>`; in particular, it could be an entirely empty relation. We demonstrate this in the theory interpretation `eandr1interp`, where all beings exist in reality, and none are `>` than any other (some may think this describes the real world), and beings are interpreted as natural numbers. PVS generates proof obligations (not shown here) to ensure the axioms of the theory `eandr1` are theorems under this interpretation, and these are trivially true.

Such a model seems contrary to the intent of the Argument: surely it is not intended that something than which there is no greater is so because nothing is greater than anything else. So we should surely require some minimal constraint on `>` to eliminate such vacuous models. A plausible constraint is that `>` be trichotomous; if we add this condition, as in `Greater1_circ_alt1`, then the premise `Greater1` can again be proved from the conclusion `God_re_alt`. (Note that the string `IMPLIES` and the symbol `=>` are synonyms in PVS, we alternate them for readability.) A weaker condition is to require only that beings satisfying the `God?` predicate should stand in the `>` relation to others; this is stated in `Greater1_circ_alt2` and is also sufficient to prove `Greater1` from `God_re_alt`.

In terms of the abstract formulation given at the beginning of Section 2, what we have here is that the conclusion C can be proved using the questionable premise Q: $P, Q \vdash C$, but not *vice versa*. However, if we augment the other premises P by adding some P_2, then we can indeed prove Q: $P, P_2, C \vdash Q$, and also the equivalence of C and Q: $P, P_2 \vdash C = Q$. Thus, Q does not beg the question C under the original premises P but does do so under the augmented premises P, P_2. We will say that Q *weakly begs* the question, where P_2 determines the "degree" of weakness.

In this example, the question begging premise fails our definition of strict begging because it is used in an impoverished theory, and weak begging compensates for that. Another way a premise can escape strict begging is by being stronger than necessary and one way to compensate for that is to strengthen the conclusion by conjoining some S so that $P, (C \wedge S) \vdash Q$ and $P, Q \vdash (C \wedge S)$. However, it may be difficult to satisfy both of these simultaneously and the first is equivalent to weak begging with $P_2 = S$; hence, we prefer the original, more versatile, notion of weak begging.

Observe that one can always construct a P_2 and thereby claim weak begging; the question is whether it is plausible and innocuous in the intended interpretation, and this is a matter for human judgment.

Addendum to the published paper. I should have observed that `ExUnd` in this example satisfies the criterion for strict begging, but it is obviously unreasonable to accuse it of begging the question because it is needed to supply the witness for `x` in the conclusion. My formal definitions for question begging are not unequivocal: they identify candidates that might be considered to beg the question, but human judgment is required to decide the matter.

4 Indirectly Begging the Question

Eder and Ramharter consider `Greater1` an unsatisfactory premise because it does not express "conceptions presupposed by the author" (i.e., Anselm) [12, Section 3.2] and says nothing about what it means to be *greater* other than the contrived connection to *exists in reality*. They propose an alternative premise `Greater2`, which is shown in Figure 4. This theory is the same as that of Figure 3, except that `Greater2` is substituted for `Greater1`, and a new premise `Ex_re` is added.

```
eandr2: THEORY
BEGIN

  beings: TYPE
  x, y: VAR beings

  >(x, y): bool

  God?(x): bool = NOT EXISTS y: y > x

  re?(x): bool

  ExUnd: AXIOM EXISTS x: God?(x)

  Ex_re: AXIOM EXISTS x: re?(x)

  Greater2: AXIOM FORALL x, y: (re?(x) AND NOT re?(y) => x > y)

  God_re_alt: THEOREM EXISTS x: God?(x) AND re?(x)

END eandr2
```

Figure 4: Eder and Ramharter's Adjusted First Order Treatment, in PVS

It is easy to prove the conclusion `God_re_alt` from the new premises; they also directly entail `Greater1` so there is circumstantial evidence that they are question begging. However, it is not possible to prove `Greater2` from `God_re_alt` and the other premises, nor have I found a plausible augmentation to the premises that enables this. Thus, it seems that `Greater2` does not beg the question under our current definitions, neither strictly nor weakly, so we should investigate whether some alternative method might expose it to this charge.[3]

When constructing a mechanically checked proof of `God_re_alt` using `Greater2` I was struck how neatly the premise exactly fits the requirement of the interactive proof at its penultimate step. To see this, observe the PVS sequent shown below; we arrive at this point following a few straightforward steps in the proof of `God_re_alt`. First, we introduce the premises `ExUnd` and `Ex_re`, expand the definition of `God?`, and perform a couple of routine steps of Skolemization, instantiation, and propositional simplification.

PVS Sequent A

```
God_re_alt :

[-1]  re?(x!1)
  |-------
{1}   x!1 > x!2
[2]   re?(x!2)
```

PVS represents its current proof state as the leaves of a tree of sequents (here there is just one leaf); each sequent has a collection of numbered formulas above and below the `|---` turnstile line; the interpretation is that the conjunction of formulas above the line should entail the disjunction of those below. Bracketed numbers on the left are used to identify the lines, and braces (as opposed to brackets) indicate this line is new or changed since the previous proof step. Terms such as `x!1` are Skolem constants. PVS eliminates top level negations by moving their formulas to the other side of the turnstile. Thus the sequent above is equivalent to the following.

Variant Sequent

```
God_re_alt :

[-1]  re?(x!1)
[2]   NOT re?(x!2)
  |-------
{1}   x!1 > x!2
```

We can read this as

[3] **Addendum** to the published version: as noted in the previous addendum, `ExUnd` is strictly begging here, and so is `Ex_re`, but human judgment acquits them of truly begging the question.

```
re?(x!1) AND NOT re?(x!2) IMPLIES x!1 > x!2
```

and then observe that `Greater2` is its universal generalization.

PVS has capabilities that help mechanize this calculation. If we ask PVS to generalize the Skolem constants in the original sequent, it gives us the formula

```
FORALL (x_1, x_2: beings): re?(x_2) IMPLIES x_2 > x_1 OR re?(x_1)
```

Renaming the variables and rearranging, this is

```
FORALL (x, y: beings): (re?(x) AND NOT re?(y)) IMPLIES x > y
```

which is identical to `Greater2`. Thus, `Greater2` corresponds *precisely* to the formula required to discharge the final step of the proof.

I will say that a premise *indirectly* begs the question if it supplies exactly what is required to discharge a key step in the proof. Unless they are redundant or superfluous, all the premises to a proof will be essential to its success, so it may seem that any premise can be considered to indirectly beg the question. Furthermore, if we do enough deduction, we can often arrange things so that the final premise to be installed exactly matches what is required to finish the proof. My intent is that the criterion for indirect begging applies only when the premise in question perfectly matches what is required to discharge a key (usually final) step of the proof when the preceding steps have been entirely routine. It is up to the individual to decide what constitutes "routine" deduction; I include Skolemization, propositional simplification, definition expansion and rewriting, but draw the line at nonobvious quantifier instantiation. The current example does require quantifier instantiation: a few steps prior to Sequent A above, the proof state is represented by the following sequent.

PVS Sequent

```
God_re_alt :

{-1}  God?(x!1)
{-2}  re?(x!2)
  |-------
[1]   EXISTS x: God?(x) AND re?(x)
```

The candidates for instantiating `x` are the Skolem constants `x!1` or `x!2`. The correct choice is `x!1` and I would allow this selection, or even some experimentation with different choices, within the "obvious" threshold, though others may disagree.

I claim that the sequent constructed by the PVS prover following routine deductions is a good representation of our epistemic state after we have digested the other premises. If the questionable premise then supplies exactly what is required to complete the proof (by generalizing the sequent), then it appears reverse-engineered, and certainly eliminates any hope of surprise or insight. Hence, I consider it to beg the question.

My description of indirect begging is very operational and might seem tied to the particulars of the PVS prover, so we can seek a more abstract definition. After we have installed the other premises, the PVS sequent is a representation of $P \supset C$. The proof engineering that reveals Q indirectly to beg the question shows that Q is what is needed to make this a theorem, so $\vdash Q \supset (P \supset C)$. But more than this, it is *exactly* what is needed, so we could suppose $\vdash Q = (P \supset C)$ and then take this as a definition of indirect begging.[4] Notice that strict begging implies this definition, but not vice-versa. However, a difficulty with this definition is that the direction $\vdash (P \supset C) \supset Q$ is generally stronger than can be proved. The proof engineering approach to indirect begging can be seen as an operational way to interpret and approximate this definition: we use deduction to simplify $P \supset C$ and then ask whether Q is its universal generalization.

In simple cases, the proof engineering approach is straightforward and makes good use of proof automation, but it may be difficult to apply in more complex proofs where a premise is employed as part of a longer chain of deductions. In the following section I show how careful proof structuring can, without undue contrivance, isolate the application of a premise and expose its question begging character.

Addendum to the published paper. Again, I failed to provide an intuitive explanation why `Greater2` should be considered to beg the question. To see this, note that we can weaken its formula by existentially quantifying the `x` on the right hand side of the implication; but then that right hand side is just `NOT God?(y).` We can then do some propositional rearrangement of the formula to yield `Greater2_circ` as shown below.

```
Greater2_circ: COROLLARY re?(x) AND God?(y) => re?(y)
```

This surely begs the question, for it says that the other two premises (which supply `re?(x)` and `God?(y)`) directly imply the conclusion. The original `Greater2` is simply a strengthened and obfuscated version of `Greater2_circ` and inherits its question begging character. In fact, I would argue the general case: any formula that is readily seen to entail an obviously question begging premise should itself be considered question begging. In that regard, observe that `Ex_re?` and `Greater2` together entail `Greater1`, which we have already established to be question begging. This

[4] I am grateful to one of the reviewers for this suggestion.

provides another intuitive reason why `Greater2` should be considered to beg the question.

5 Indirect Begging in More Complex Proofs

In search of a more faithful reconstruction of Anselm's Argument, Eder and Ramharter observe that Anselm attributes properties to beings and that some of these (notably *exists in reality*) contribute to evaluation of the *greater* relation [12, Section 3.3]. They formalize this by hypothesizing some class `P` of "*greater*-making" properties on beings and then define one being to be greater than another exactly when it has all the properties of the second, and more besides. This treatment is higher order because it involves quantification over properties, not merely individuals. This is seen in the definition of `>` in the PVS formalization of Eder and Ramharter's higher order treatment shown in Figure 5. Notice that `P` is a set (which is equivalent to a predicate in higher-order logic) of predicates on `beings`; in PVS a predicate in parentheses as in `F: VAR (P)` denotes the corresponding subtype, so that `F` is a variable ranging over the subsets of `P`. A more detailed description of this PVS formalization is provided elsewhere [26].

The strategy for proving `God_re_ho` is first to consider the being `x` introduced by `ExUnd`; if this being exists in reality, then we are done. If not, then we consider a new being that has exactly the same properties as `x`, plus existence in reality—this is attractively close to Anselm's own strategy, which is to suppose that very same being can be (re)considered as existing in reality. In the PVS proof this is accomplished by the proof step

PVS Proof Step
```
(name "X" "choose! z: FORALL F: F(z) = (F(x!1) OR F=re?)")
```

which names `X` to be such a being. Here, `x!1` is the Skolem constant corresponding to the `x` introduced by `ExUnd` and `choose!` is a "binder" derived from the PVS choice function `choose`, which is defined in the PVS Prelude. This `X` is some being that satisfies all the predicates of `x!1`, plus `re?`. Given this `X`, we can complete the proof, except that PVS generates the subsidiary proof obligation shown below to ensure that the choice function is well-defined (i.e., there is such an `X`).[5]

PVS TCC
```
EXISTS (x: beings): (FORALL F: F(x) = (F(x!1) OR F = re?))
```

[5] This is similar to the proof obligation generated for the definite description used in Oppenheimer and Zalta's rendition: there we had to prove that the predicate in `the` is uniquely satisfiable; here we need merely to prove that the predicate in `choose!` is satisfiable. The properties of the definite description, the choice function, and Hilbert's ε are described and compared in our description of Oppenheimer and Zalta's treatment [25].

```
eandrho: THEORY
BEGIN

  beings: TYPE

  x, y, z: VAR beings

  re?: pred[beings]

  P: set[ pred[beings] ]

  F: VAR (P)

  >(x, y): bool = (FORALL F: F(y) => F(x)) & (EXISTS F: F(x) AND NOT F(y))

  God?(x): bool = NOT EXISTS y: y > x

  ExUnd: AXIOM EXISTS x: God?(x)

  Realization: AXIOM
    FORALL (FF:setof[(P)]): EXISTS x: FORALL F: F(x) = FF(F)

  God_re_ho: THEOREM member(re?, P) => EXISTS x: God?(x) AND re?(x)

END eandrho
```

Figure 5: Eder and Ramharter's Higher Order Treatment, in PVS

This proof obligation requires us to establish that there is a being that satisfies the expression in the `choose!`; it is generated from the predicate subtype specified for the argument to `choose` and is therefore called a PVS Typecheck Correctness Condition, or TCC [30].[6]

Eder and Ramharter provide the axiom `Realization` for this purpose; it states that for any collection of properties, there is a being that exemplifies *exactly* those properties and, when its variable `FF` is instantiated with the term

`{G: (P) | G(x!1) OR G=re?}`,

[6]**Addendum** to the published paper. PVS 7.1 rationalized the presentation of TCCs, so the one shown is now replaced by the following (equivalent) one.

PVS TCC
```
nonempty?[beings](LAMBDA z: FORALL F: F(z)=(F(x!1) OR F=re?))
```

it provides exactly the expression above. In other words, `Realization` is a generalization of the formula required to discharge a crucial step in the proof. Thus, I claim that the premise `Realization` indirectly begs the question in this proof. This seems appropriate to me, because `Realization` says we can always "turn on" real existence and, taken together with `ExUnd` and the definition of `>`, this amounts to the desired conclusion.

An alternative and more common style of proof in PVS would invoke the premise `Realization` directly at the point where `name` and `choose!` are used in the proof described here. The direct invocation obscures the relationship between the formal proof and Anselm's own strategy, and it also uses `Realization` as one step in a chain of deductions that masks its question begging character. Thus, use of `name` and `choose!` are key to revealing both the strategy of the proof and the question begging character of `Realization`. Note that the deductions prior to the `name` command, and those on the subsequent branch to discharge the TCC should be routine if `Realization` is to be considered indirectly question begging, but those on the other branch may be arbitrarily complex.

Addendum to the published paper. If we perform the substitution mentioned above for the variable `FF` in `Realization` and do some deduction, then we arrive at the following formula.

```
Greater_triv1: LEMMA member(re?, P) => FORALL x: re?(x) OR EXISTS y: y>x
```

The right hand disjunct is just `NOT God?(x)`, and then some propositional rearrangement gives us the following.

```
Greater_triv2: LEMMA member(re?, P) => FORALL x: God?(x) => re?(x)
```

But this clearly begs the question: it takes us straight from `ExUnd` to the conclusion `God_re_ho`. Since `Realization` readily entails this formula, we have an intuitive reason why it should be considered question begging.
End of addendum

Campbell [8], who is completing a new book on the Argument [7], adopts some of Eder and Ramharter's higher order treatment, but rejects `Realization` on the grounds that it is false. Observe that we could have incompatible properties[7] and

[7]Eder and Ramharter are careful to require that all the greater-making properties are "positive" so directly contradictory properties are excluded, but we can have positive properties that are mutually incompatible [16]. Examples are being "perfectly just" and "perfectly merciful": the first entails delivering exactly the "right amount" of punishment, while the latter may deliver less than is deserved.

`Realization` would then provide the existence (in the understanding) of a being that exemplifies those incompatible properties, and this is certainly questionable. A better approach might be to weaken `Realization` to allow merely the addition of `re?` to the properties of some existing being. This is essentially the approach taken below.

```
campbell: THEORY
BEGIN

  beings: TYPE

  x, y, z: VAR beings

  re?: pred[beings]

  P: set[ pred[beings] ]

  F: var (P)

  >(x, y): bool = (FORALL F: F(y) => F(x)) & (EXISTS F: F(x) AND NOT F(y))

  God?(x): bool = NOT EXISTS y: y > x

  ExUnd: AXIOM EXISTS x: God?(x)

  quasi_id(D: setof[(P)])(x,y: beings): bool =
     FORALL (F:(P)): NOT D(F) => F(x) = F(y)

  jre: setof[(P)] = singleton(re?)

  Weak_real: AXIOM
      NOT re?(x) => (EXISTS z: quasi_id(jre)(z, x) AND re?(z))

  God_re_ho: THEOREM member(re?, P) => EXISTS x: God?(x) AND re?(x)

END campbell
```

Figure 6: Simplified Version of Campbell's Treatment, in PVS

Campbell's formal treatment [8] differs from others considered here in that he includes more of Anselm's presentation of the Argument (e.g., where he speaks of "the Fool"). The treatment shown in Figure 6 is my simplified interpretation of Campbell's approach, scaled back to resemble the other treatments considered. Campbell adopts Eder and Ramharter's higher order treatment, but replaces `Realization` by (in my interpretation) the axiom `Weak_real` which essentially states that if `x` does not exist in reality, then we can consider a being just like it that does. A being "just like it" is defined in terms of a predicate `quasi_id` introduced by Eder and Ramharter [12, Section 3.3] and is true of two beings if they have the same properties, except possibly those in a given set `D`. Observe that the PVS specification writes this higher order predicate in Curried form. Here, `D` is always instantiated by the singleton set `jre` containing just `re?`, so we always use `quasi_id(jre)`.

A couple of routine proof steps bring us to the following sequent.

```
God_re_ho :

{-1}  P(re?)
  |-------
[1]   EXISTS y: y > x!1
[2]   re?(x!1)
```

Our technique for discharging this is to instantiate formula 1 with a being just like `x!1` that does exist in reality, which we name `X`.

```
  (name "X" "(choose! z: quasi_id(jre)(z, x!1) AND re?(z))")
```

The main branch of the proof then easily completes and we are left with the obligation to ensure that application of the choice function is well-defined. That is, we need to show

```
EXISTS (z: beings): quasi_id(jre)(z, x!1) AND re?(z)
```

under the condition `NOT re?(x!1)`. This is precisely what the premise `Weak_real` supplies, so we may conclude that this premise indirectly begs the question.

Addendum to the published paper. The premise `Weak_real` readily entails `Greater_triv2` just as `Realization` does, and so provides an intuitive reason why it, too, should be considered question begging.
End of addendum

The higher order formalizations considered in this section have slightly longer and more complex proofs than those considered earlier. This means that the indirect question begging character of a particular premise may not be obvious if it occurs in the middle of a chain of proof steps. Use of the `name` and `choose!` constructs accomplishes two things: it highlights the strategy of the proof (namely, it identifies the attributes of the alternative being to consider if the first one does not exist in reality), and it isolates application of the questionable premise to a context where its indirect question begging character is revealed.

6 Comparison with Informal Accounts of Begging the Question

There are several works that examine the Ontological Argument against the charge that it begs the question. Some of these, including the present paper, employ a "logical" interpretation for begging the question, which is to say they associate question begging with the logical form of the argument and not with the meaning attached to its symbols. Others employ a "semantical" interpretation and find circularities in the meanings of the concepts employed by the Argument prior to consideration of its logical form.

Roth [23], for example, observes that Anselm begins by offering a definition of God as that than which nothing greater can be conceived and then claims that greatness already presupposes existence and is therefore question begging. McGrath [19] criticizes Rowe's analysis and presents his own, which finds circularity in the relationship between possible and real existence. (Kant, who named the Argument, declared that existence is not a predicate [17].) Devine [10] (who was writing 15 years earlier than McGrath but is not cited by him) asks whether it is possible to use "God" in a true sentence without assuming His existence and concludes that it is indeed possible and thereby acquits the Argument of this kind of circularity.

All these considerations lie outside the scope considered here. We treat "greater than," "real existence," and any other required terms as uninterpreted constants, and we assume there is no conflict between the parts they play in the formalized Argument and the intuitive interpretations attached to them. We then ask whether the formalized argument begs the question in a logical sense.

Many authors consider logical question begging in semi-formal arguments. Some consider a "dialectical" interpretation associated with the back and forth style of argumentation that dates to Aristotle's original identification of the fallacy (as he thought of it), while others consider an "epistemic" interpretation in the context of standard deductive arguments. Walton [36] outlines a history of analysis of begging the question, focusing on the dialectical interpretation, while Garbacz [13] provides a formal account within this framework. Walton [34] contends that the notion of question begging and the intellectual tools to detect it are similar in both the dialectical and epistemic interpretations, so I will focus on the epistemic case. The intuitive idea is that a premise begs the question epistemically when "the arguer's belief in the premise is dependent on his or her reason to believe the conclusion" [34, page 241].

Several authors propose concrete definitions or methods for detecting epistemic question begging. Walton [34], for example, recommends proof diagrams (as supported in the Araucaria system [22]) as a tool to represent the structure of informal arguments, and hence reveal question begging circularities. He illustrates this with "The Bank Manager Example":

Manager: Can you give me a credit reference?

Smith: My friend Jones will vouch for me.

Manager: How do we know he can be trusted?

Smith: Oh, I assure you he can.

Our interest here is with formal arguments and as soon as one starts to formalize The Bank Manager Example, it becomes clear that the argument is invalid, for it has the following form.

Premise 1: $\forall a, b : \texttt{trusted}(a) \wedge \texttt{vouch-for}(a, b) \supset \texttt{trusted}(b)$

Premise 2: $\texttt{vouch-for}(\texttt{Jones}, \texttt{Smith})$

Premise 3: $\texttt{vouch-for}(\texttt{Smith}, \texttt{Jones})$

Conclusion: $\texttt{trusted}(\texttt{Smith})$

The invalidity here is stark and independent of any ideas about question begging. Walton describes other methods for detecting question begging in informal arguments but most of the examples are revealed as invalid when formalized. While these methods may be of assistance to those committed to notions of informal argument or argumentation, our focus here is on valid formal arguments, so we do not find these specific techniques useful, although we do subscribe to the general "epistemic" model of question begging, and will return to this later.

Barker [4], building on [3, 32], calls a deductive argument *simplistic* if it has a premise that entails the conclusion; he claims that all and only such (valid) arguments are question begging. Our definition for strict begging includes this case, but also others. For example, Barker considers the argument with premises p and $\neg q$ and conclusion p to beg the question, whereas that with premises $p \vee q$ and $\neg q$, and the same conclusion does not, which seems peculiar to say the least. Both of these are question begging by our strict definition.

Now one might try to "mask" the question begging character of an argument that satisfies Barker's definition by adding obfuscating material, so he needs some notion of equivalence to expose such "masked" arguments. However, it cannot be logical equivalence of the premises because the conjunction of premises is identical in the two cases above, yet Barker considers one to be question begging and the other not. Barker proposes that "relevant equivalence" (i.e., the bidirectional implication of relevance logic [11]) of the premises is the appropriate notion. The examples above are not equivalent by this criterion ($\neg q \supset p$ and $\neg q$ illustrate premises that are equivalent to the second example by this criterion) and so the question begging character of the first does not implicate the second, according to Barker.

As noted, all these examples strictly beg the question by my definition and I claim this is as it should be. Recall that a premise strictly begs the question when it is equivalent to the conclusion, given the other premises. Now, the essence of the epistemic interpretation for begging the question is that truth of the premise in question is difficult to know or believe independently of the conclusion, and I assert that this judgment must be made after we have digested the other premises (otherwise, what is their purpose?). Thus, if $\neg q$ is given (digested), then $p \vee q$ and p are logically equivalent and we cannot believe one independently of the other and $p \vee q$ is rightly considered to beg the question in this context. Barker judges $p \vee q$ and p in the absence of any other premise and thereby reaches the wrong conclusion, in my opinion.

My proposal for strict begging differs from those in the literature but is not unrelated to existing proposals such as Barker's. My proposals for weak and indirect begging depart more radically from previous treatments. I consider a premise to be weakly begging when light augmentation to the other premises render it strictly begging. Human judgment must determine whether the augmentation required is innocuous or contrived and this can be guided by epistemic considerations: if the augmentation is required to establish a context in which the questionable premise(s) are plausible (as in our example of Figure 3, where we certainly intend the $>$ relation to be nonempty), then the questionable premise(s) surely beg the question in the informal epistemic sense as well as in our formal weak sense.

Indirect begging arises when the questionable premise supplies (a generalization of) exactly what is required to make a key move in the proof. Provided we have not applied anything beyond routine deduction, I claim that the proof state (conveniently represented as a sequent) represents our epistemic state after digesting the other premises and the desired conclusion. An indirectly begging premise is typically (a generalization of) one that can be reverse engineered from this state, and belief in such a premise cannot be independent of belief in the current proof state; hence such a premise begs the question in the informal epistemic sense as well as in our formal indirect sense.

The informal epistemic criterion underpins our definitions for begging the question in formal deductive arguments. These identify when a premise may be considered to beg the question, but it is not immediate from these definitions why this should be considered a defect. The conclusion to a deductive argument is always implicit or "contained" in the premises but one source of value or satisfaction can be surprise at the revelation that the premises do indeed entail the conclusion. This is surely one reason for the enduring interest in the Ontological Argument: its premises seem innocuous, yet its conclusion is bold. But when a premise is shown to beg the question, this surprise is seen to be illusory: we already assented to the conclusion when we accepted the premise in question.

Most authors who examine question begging in the Ontological Argument implicitly apply an epistemic criterion, and do so in the context of modal representations of the argument (which are briefly mentioned below). Walton, however, does discuss first-order formulations in a paper that is otherwise about modal formulations [33].

Walton begins with a formulation that is identical (modulo notation) to that of Figure 4. He asserts that the premise `Greater2` (his premise 2) is implausibly strong because it "would appear to imply, for example, that a speck of dust is greater than Paul Bunyan."[8] I would suggest that a better indicator of its "implausible strength" is the fact that it indirectly begs the question, as described in Section 4. Walton then proposes that premise `Greater1` of Figure 3 (his premise 2G) may be preferable but worries that our reason for believing `Greater1` must be something like `Greater2`. It is interesting that Walton does not indicate concern that `Greater1` might beg the question, whereas our analysis shows that it is weakly begging, and becomes strictly so in the presence of premises that require a modicum of connectivity in the `>` relation (recall Sections 2 and 3). Thus, I suggest that the formulations and methods of analysis proposed here are more precise, informative, and checkable than Walton's and other informal interpretations for begging the question.

7 Conclusion

Once we go beyond the "simplistic" case, where the conclusion is directly entailed by one of its premises, the idea of begging the question is open to discussion and personal judgment. A variety of positions are contested in the literature on argumentation and were surveyed in Section 6, but I have not seen any discussion of question begging in fully formal deductive settings.

My proposal is that a premise may be considered to beg the question when it is equivalent to the conclusion, given the other premises (strict begging), or a light

[8]Paul Bunyan is a lumberjack character in American folklore.

augmentation of these (weak begging), or when it directly discharges a key step of the proof (indirect begging). The intuition is that such premises are so close to the conclusion or its proof that they cannot be understood or believed independently of it. I have shown that several first- and higher-order formalizations of the Ontological Argument beg the question, illustrating each of the three kinds of question begging. I suspect that all similar formulations of the Argument are vulnerable to the same charge.

Separately (in work performed after this paper was prepared) [28], I have examined several formulations of the argument in quantified modal logic (including that of Rowe [24], who explicitly accuses the Argument of begging the question, and those of Adams [1] and Lewis [18], who also discuss circularity) and found them vulnerable to the same criticism. The analysis there reveals that modal formulations of the Argument admit delicate choices in how the quantification is arranged and this determines identification of the premises accused of question begging.

Begging the question is not a fatal defect and does not affect validity of its argument; identification of a question begging premise can be an interesting observation in its own right, as may be identification of the augmented premises that reveal a weakly begging one. However, I think most would agree that the persuasiveness of an argument is diminished when its premises are shown to beg the question. Furthermore, revelation of question begging undermines any delight or surprise in the conclusion, for the question begging premise is now seen to express the same idea.

Indirect begging is perhaps the most delicate case: it reveals how exquisitely crafted—one is always tempted to say reverse-engineered—is the questionable premise to its rôle in the proof. To my mind, it casts doubt on the extent to which the premise may be considered analytic in the sense that Eder and Ramharter use the term: that is, something that the author "could have held to be true for conceptual (non-empirical) reasons" [12, Section 1.2(7)].

In a related observation, Eder and Ramharter note that in a deductively valid argument the premises always "contain" the conclusion but, for an argument to be satisfying, they should do so in a *non-obvious* way: the conclusion has to be "hidden" in the premises [12, Section 1.2(5)]. One way of looking at the notions of question begging defined here is that they identify cases where the conclusion is insufficiently well hidden. A legitimate criticism is that the methods employed, particularly for weak and indirect begging, may be too powerful, so that intuitively "well hidden" conclusions are exposed by unreasonably intense scrutiny. Richard Campbell expresses this concern [9] and poses an example derived from Proslogion III. Here, we have two premises

1. "Something-than-which-a-greater-cannot-be-thought" (STWNG) so truly exists that it cannot be thought not to exist.

2. Whatever is other than God can be thought not to exist.

The desired conclusion is "God is STWNG" (and hence exists).

Since this has only two premises, if we are given either one plus the conclusion it is always possible to calculate the other, and Campbell is concerned this can be used to justify an accusation of indirect begging. He finds this argument to be a satisfying one since the first premise says nothing about God, and the second says nothing about His greatness, so the conclusion is nicely hidden in the premises.

This is a modal argument (i.e., it involves *necessary* and *possible* existence) and I prefer not to complicate this paper with a description of how modal arguments are embedded in PVS (this is done at length elsewhere [27, 28]), but the salient point is that my methods can indeed be used unjustly to accuse this argument of begging the question.

First, we might claim that STWNG should be unique under the intended interpretation. If we add this as a premise, then Premise 2 can be proved from the other premises and the Conclusion and is therefore weakly begging. Separately, Premise 1 can be reverse-engineered (and thereby claimed as indirectly begging) from the Conclusion and Premise 2, but the derivation involves a quantifier instantiation (of STWNG for the "whatever" variable in Premise 2).

Now, weak begging is "graduated" by the strength of the augmenting premise, and indirect begging by the deductive power employed, so this example nicely illustrates the range of judgments that are possible. Campbell states that to augment the premises is not merely unnecessary but an error, for uniqueness is a consequence, not an assumption, of this argument. Furthermore, the quantifier instantiation required to exhibit indirect begging is not routine (indeed, PVS does not find it automatically) but a creative step. Thus, the accusations of weak and indirect begging should both be rejected in this example. In contrast, the augmentation and deductive power needed to reveal weak and indirect begging in the Proslogion II argument seem reasonable to me and serve correctly to identify the premises concerned as contrived rather than analytic.

It is, of course, for individual readers to form their own opinions and to decide whether the forms of question begging identified here affect their confidence, or their interest, in the various renditions of Anselm's Argument, or in the Argument itself. What I hope all readers find attractive is that these methods provide explicit evidence to support accusations of question begging that can be exhibited, examined, and discussed, and that may be found interesting or enlightening even if the accusations are ultimately rejected.

Observe that detection of the various kinds of question begging requires exploring variations on a specification or proof. This is tedious and error-prone to do by hand, but simple, fast, and reliable using mechanized assistance. I hope the methods and tools illustrated here will encourage others to investigate similar questions concerning this and other formalized arguments: as Leibniz said, "let us calculate."

Addendum to the published paper. As noted in the Addendum to Section 2, the axiom `Greater1` of Figures 2 and 3 can be replaced by `Greater1cp_alt`. But then the corresponding theorems can be proved without opening the definition of `God?`. In Figure 2, the definition of `God?` is also used to discharge the proof obligation that establishes uniqueness of its definite description, but in Figure 3 it would not be used at all. Hence, we can attach any interpretation we like to this predicate, including Gaunilo's "most perfect island" [15] and so this formalization of the argument is essentially vacuous. The same criticism applies to all the formal treatments considered later in the paper (in particular, each of them entails an obviously question begging formula that is derived in its Addendum, and these can all prove the conclusion without opening their definition for God). This topic is discussed in more detail in a subsequent paper [28].

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