On First Order Interval Temporal Logic

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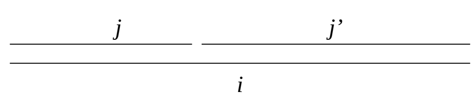
Chapter 1

Introduction

Digital systems are increasingly used in applications where they interact with physical processes. These systems often have to meet real-time constraints: they have to react to events within a prescribed time interval, to produce output before a certain delay has elapsed, etc. In order to reason about such real-time applications, quantitative as well as qualitative time requirements have to be considered. For this purpose, various real-time temporal logics have been proposed.

For example, several real-time extensions of linear propositional temporal logic (PTL) are reviewed and compared in [4]. Although these logics are substantially more complex than ordinary PTL, some of them conserve interesting properties such as decidability [4]. Similarly, real-time extensions of the branching time logic CTL have been introduced [10, 19] for which model checking is decidable [3, 17].

In the above logics, formulas are interpreted over states which represent instantaneous situations; time points are the basic entities. Other formalisms adopt a different semantics and interpret formulas over intervals of time [20, 12, 27]. Among such interval modal logics, ITL [20] and more specifically the duration calculus [8, 24] have been proposed for reasoning about real-time systems. These two formalisms are first order modal logics which incorporate a binary modal operator (denoted by ';') interpreted as the operation of 'chopping' an interval into two parts: a formula $(f; g)$ is satisfied by an interval i if i can be split into two sub-intervals j and j' as follows



with j satisfying f and j' satisfying g.

Other systems of temporal logics also incorporate the operator chop. It is known that the addition of chop and of its reflexive and transitive closure to PTL yields a logic which has the same expressive power as full regular expressions [25]. A decision procedure and a complete proof system for such a propositional logic are given in [25]. Other complete deductive systems for propositional modal logics which include the operator chop can also be found in [23] and [27].

In the first order case, different deductive systems exist for both ITL [21] and the duration calculus [14, 26] but little is known about their power. Close links between the two logics have been established in [14]: a complete proof system for a dense-timed ITL would yield a complete deductive system for the duration calculus. Except for restricted fragments, the duration calculus (and ITL) are not decidable [7].

In this document, we examine completeness problems for first order ITL in a variant similar to the one introduced in [14] which contains no other modal operator than chop[1]. We consider different classes of models of the logic and we give a complete and sound proof system for each class.

- ◇ First, we give *possible world* semantics to ITL which generalizes the traditional interval-based semantics. We define a first proof system S adequate for a class of possible worlds models. Completeness is shown by using classic techniques, similar to those presented in [11] and [1]. The main interest of this first result is to provide a general model construction which can be applied to any consistent extension of S.

- ◇ Then, we concentrate on *interval models* similar to the traditional ones. These are constructed from a notion of linear temporal domain together with a measure function which assigns a length to intervals. In this context, real-time properties can be expressed as relations on the length or duration of intervals. We give a deductive system for reasoning about such interval models and we show that this system is complete. The proof of completeness uses the general model construction developed for S and a translation from possible worlds to interval models.

In the remainder of this document, chapter 2 presents the syntax and possible world semantics of first order interval temporal logic. Chapter 3 describes the deductive system S and the associated class of models, and exposes the first completeness result. Chapter 4 is dedicated to interval models. A proof system S' is defined and S' is shown to be sound and complete for the class of interval models. In chapter 5 a few applications of the completeness results are exposed. Several extensions of S' are considered which make various assumptions on time or the properties of models and the problem of expressing finite variability in ITL is examined.

[1] Other modalities such as $\Box$ (in all sub-intervals) or $\Diamond$ (in some sub-interval) can still be easily defined in terms of chop (see [14] for example).

Chapter 2

First order ITL

2.1 Syntax

2.1.1 Language

A language for first order ITL with equality (or ITL-language) consists of a denumerable collection of function and predicate symbols. With each symbol is associated an non-negative integer as its arity. Predicate symbols of arity 0 are propositions and function symbols of arity 0 are individual constants.

In addition, we distinguish between flexible and rigid symbols (we use the terminology of [1, 11]). Rigid symbols are intended to represent fixed, global entities. Their interpretation will be the same in all the intervals or worlds of a model. Conversely, entities which may vary in different intervals or worlds are represented by flexible symbols.

Such a distinction between two classes of symbols is common in the context of first order temporal logics [1, 9]. It also appears in the duration calculus and ITL although it is often restricted to propositions and individual constants only; all the functions and predicates of non-null arity are considered rigid [21, 14, 8]. In order to be as general as possible, we do not make such a restriction, function and predicate symbols of any arity can be flexible.

An ITL-language specifies a set of non-logical symbols from which terms and formulas are constructed. The vocabulary also contains an infinite, denumerable set of variables $V = \{x_1, x_2, \ldots\}$, the existential quantifier $\exists$, the connectives $\wedge$ and $\neg$, and the symbol '$=$' and a single binary modal connectives '$;$'. The equality symbol is considered as a supplementary *rigid* binary predicate.

2.1.2 Terms

For a fixed language $\mathcal{L}$, the set of terms is defined – as in ordinary first order logic – as the smallest set which satisfies the following rule:

- $\diamond$ any variable x_i is a term,
- $\diamond$ any constant a is a term,

- if $t_1, \ldots, t_n$ are n terms $(n > 0)$ and α a function symbol of arity n then $\alpha(t_1, \ldots, t_n)$ is a term.

We say that a term t is *flexible* if it contains some flexible constant or function symbol of $\mathcal{L}$. Conversely, a term in which no flexible symbol occurs is said to be *rigid*. In particular all the variables are rigid.

2.1.3 Formulas

Atomic formulas are also defined as in first order logic with equality. An atomic formula is either

- a propositional symbol p,
- an expression $\phi(t_1, \ldots, t_n)$ where ϕ is a predicate symbol of arity $n > 0$ and $t_1, \ldots, t_n$ are n terms, or
- an identity $(t_1 = t_2)$ where t_1 and t_2 are two terms.

The set of formulas is the smallest set which satisfies the following rules:

- any atomic formula is a formula,
- if f is a formula, then $(\neg f)$ is a formula,
- if f_1 and f_2 are formulas then $(f_1 \wedge f_2)$ and $(f_1; f_2)$ are formulas,
- if f is a formula and x a variable then $(\exists x)f$ is a formula.

The other standard logic connectives and the universal quantifier are introduced as abbreviations. If f_1 and f_2 are two formulas,

- $(f_1 \Rightarrow f_2)$ stands for $(\neg(f_1 \wedge (\neg f_2)))$,
- $(f_1 \vee f_2)$ for $((\neg f_1) \Rightarrow f_2)$, and
- $(f_1 \Leftrightarrow f_2)$ for $((f_1 \Rightarrow f_2) \wedge (f_2 \Rightarrow f_1))$.

If x is a variable and f a formula then

- $(\forall x)f$ is an abbreviation for $(\neg(\exists x)(\neg f))$.

Free and bound variables, open and closed formulas (sentences) are defined as in first order logic (see [13] for example). As for terms, we say that a formula is flexible or rigid according as whether it contains a flexible symbol or not. If a formula f does not contain the chop operator ';' then f is said to be chop-free.

In order to simplify the notations, we adopt the usual rules for suppressing excessive parentheses of logical expressions but we always keep parentheses around chop formulas. The propositional connectives have all a higher priority than ';'. For convenience, we also use infix notations for binary functional or predicate symbols such as $+$ or $\leqslant$.

2.2 Semantics

2.2.1 Models

In most of the interval logics encountered in computer science [12, 19], intervals are constructed from time points which are the primitive objects. Traditional models for ITL and the duration calculus are based on such an approach [14, 21]. We adopt a different point of view: as in [27], we define a general *possible worlds* semantics for ITL and we consider the traditional ITL models as a special cases[1]. Possible worlds models are similar the Kripke structures of classic modal logic [18]. This makes possible the application of techniques developed for showing completeness of systems of modal logic with quantifiers [11] to ITL.

Definition 2.1 *A model $\mathcal{M}$ for an ITL-language $\mathcal{L}$ is a quadruple (W, R, D, I) where*

- ◇ *W is a non-empty set of possible worlds and R a ternary accessibility relation on W,*
- ◇ *D is a non-empty set,*
- ◇ *I is a function which assigns to each symbol s of $\mathcal{L}$ and each world w in W an interpretation $I(s, w)$ as follows:*
  - *if s is an n-ary function symbol, $I(s, w)$ is a function from D^n to D,*
  - *if s is an n-ary predicate symbol, $I(s, w)$ is an n-ary relation on D,*

  and such that the interpretation of rigid symbols is the same in all worlds.

The only difference with models of classic modal logic is that the accessibility relation is ternary. The pair (W, R) is called the *frame* and D the *domain* of the model.

We consider n-ary relations as functions from D^n to $\{0, 1\}$. Functions from D^0 to any non-empty set E are identified with elements of E. Hence, for an individual constant a, $I(a, w)$ is an element of D and similarly for a proposition p, $I(p, w)$ is either 0 or 1.

2.2.2 Interpretation of terms

Given a model $\mathcal{M} = (W, R, D, I)$, a meaning is associated in each world of W to every term of $\mathcal{L}$. This meaning is an element of D and depends on particular values assigned to variables. We call an *$\mathcal{M}$-valuation* (or simply a valuation when the model considered is clear form the context) any mapping v which assigns an element of D to every variable. Given a variable x, two valuations v and v' are said to be *x-equivalent* if they agree on every variable except possibly x: for any variable y distinct from x, $v(y) = v'(y)$.

We denote by $I_w^v(t)$ the meaning of a term t in a world w under a valuation v. The function I_w^v is defined by induction on terms as follows:

[1] These will be introduced in chapter 3.

- for a constant symbol a, $I_w^v(a) = I(a, w)$,
- for a variable x, $I_w^v(x) = v(x)$,
- for a term t of the form $\alpha(t_1, \ldots, t_n)$,

$$I_w^v(t) = I(\alpha, w)(I_w^v(t_1), \ldots, I_w^v(t_n)).$$

It is clear that, for any rigid term t, $I_w^v(t)$ is the same in all the worlds w of the model.

2.2.3 Satisfaction, validity

For a formula f, we denote by $\mathcal{M}, w, v \models f$ that f is satisfied in the world w of $\mathcal{M}$ under an $\mathcal{M}$-valuation v. When there is no ambiguity about the model, we simply write $w, v \models f$.

Satisfaction in a model $\mathcal{M} = (W, R, D, I)$ is defined by the following rules:

- $w, v \models p$ iff $I(p, w) = 1$,
- $w, v \models \phi(t_1, \ldots, t_n)$ iff $I(\phi, w)(I_w^v(t_1), \ldots, I_w^v(t_n)) = 1$,
- $w, v \models t_1 = t_2$ iff $I_w^v(t_1) = I_w^v(t_2)$.
- $w, v \models f_1 \wedge f_2$ iff $w, v \models f_1$ and $w, v \models f_2$,
- $w, v \models \neg f$ iff $w, v \not\models f$,
- $w, v \models (\exists x) f$ iff there is a valuation v', x-equivalent to v, and such that $w, v' \models f$,
- $w, v \models (f_1; f_2)$ iff there are two worlds w_1 and w_2 of W such that

$$w_1, v \models f_1, \quad w_2, v \models f_2, \quad \text{and} \quad R(w_2, w_2, w).$$

Here again, it follows from the definition that, for a fixed valuation v, a rigid formula is either true in all the worlds or false in all the worlds of $\mathcal{M}$.

A model $\mathcal{M}$ *satisfies* a formula f if there is a world w of $\mathcal{M}$ and an $\mathcal{M}$-valuation v such that $\mathcal{M}, w, v \models f$. This notion extends immediately to classes of models: f is *satisfiable* in a class $\mathcal{C}$ of models if it is satisfied in some model of $\mathcal{C}$.

Given a set of formulas or sentences Γ, we say that $\mathcal{M}$ is a model of or satisfies Γ if there is a world w and a valuation v such that for every formula f of Γ, $\mathcal{M}, w, v \models f$.

A formula f is *valid in* $\mathcal{M}$ if for any world w of $\mathcal{M}$ and any $\mathcal{M}$-valuation v, $\mathcal{M}, w, v \models f$. f is *valid in a class of models* $\mathcal{C}$ if it is valid in all the members of the class, and f is *valid* if it is valid in the class of all models.

For any formula f, possibly containing free variables, it is always possible to find a sentence whose satisfiability or validity is equivalent to those of f: Let $x_1, \ldots, x_n$ be the free variables of f then

- ◇ f is satisfiable if and only if the *existential closure* $(\exists x_1)\ldots(\exists x_n)f$ is satisfiable,
- ◇ f is valid if and only if the *universal closure* $(\forall x_1)\ldots(\forall x_n)f$ is valid.

2.2.4 Examples of valid formulas

ITL can be considered as an extension of conventional first order logic with equality. The semantics ensures that any chop-free formula which is valid in first order logic is also valid in ITL. For example, if p is a unary predicate and a a constant, the following formulas are all valid:

$$p(a) \Rightarrow (\exists x)p(x), \quad (\forall x)p(x) \Rightarrow p(a), \quad \text{and} \quad x = a \wedge p(a) \Rightarrow p(x).$$

The validity of these formulas is independent of the nature of the two symbols p and a; they can be flexible or rigid.

If $p(x)$ is replaced by an arbitrary chop-free formula $f(x)$ where x is free in $f(x)$ then the resulting formulas are still valid. This is no longer true in general if $f(x)$ contains the chop connective. On the other hand it is easy to check that any ITL instance of a propositional tautology, such as

$$(p(x); q(x,a)) \wedge q(y,x) \Rightarrow (p(x); q(x,a)) \quad \text{or} \quad (p(x); r) \vee \neg(p(x); r)$$

is valid.

An important property of ITL is that chop distributes over disjunctions. For arbitrary formulas f, g, h, the two following equivalences are valid

$$(f \vee g; h) \Leftrightarrow (f; h) \vee (g; h)$$
$$(f; g \vee h) \Leftrightarrow (f; g) \vee (f; h).$$

Due to the restriction on the interpretation of rigid symbols, the satisfaction of rigid formulas is the same in all the worlds of a model. It follows that, whatever the formula g, if f is a rigid formula then both $(f; g) \Rightarrow f$ and $(g; f) \Rightarrow f$ are valid.

Finally, existential quantifiers and chop can commute under certain conditions. We have chosen a fixed domain semantics: there is only one global domain D in a model and not a domain D_w local to every world w as is sometimes done in modal logic [11, 18]. As a consequence, and because a valuation is fixed for all worlds, a variant of Barcan formula [18] holds in ITL. Formulas of the form

$$((\exists x)f(x); g) \Rightarrow (\exists x)(f(x); g) \quad \text{and} \quad (g; (\exists x)f(x)) \Rightarrow (\exists x)(g; f(x))$$

are valid, provided x is not free in g.

The converse implications are also valid, as well as, more generally, the formula

$$(\exists x)(f(x); g(x)) \Rightarrow ((\exists x)f(x); (\exists x)g(x)).$$

Chapter 3

A first axiomatic system

In this chapter we define a first deductive system S for ITL. This system will be the most general presented in this document. All the other proof systems will be extensions of S. The system S is intended to allow reasoning about a general class $\mathcal{C}$ of models which contains all the traditional interval models. We will show that S is adequate (i.e. sound and complete) for this purpose.

We first give the definition of $\mathcal{C}$ and of the proof system S, then we present several examples of derivations of theorems in S, finally we prove that S is complete.

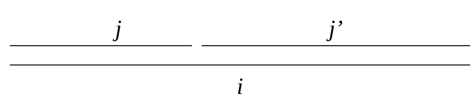
3.1 The system S

3.1.1 Models for S

In a logic such as ITL, reasoning about qualitative properties of real-time systems is based on a predefined measure or length of time intervals. This requires the presence in the language of some symbolic representation of the length. In the duration calculus, a particular symbol ℓ is provided for this purpose [14]. We adopt the same convention: from now on, all the ITL-languages considered contain at least the flexible individual constant ℓ.

Of course, a function assigning a length to different intervals is not arbitrary. For example, it might seems reasonable to assume that the length of an interval i is larger than the length of any of its sub-intervals. We will formalize some of these assumptions in chapter 4 but first we consider the following property.

Assume an interval i can be split into a prefix interval j and a suffix interval j' as follows



then the pair (j, j') is uniquely determined by either the length of j or the length of j'. If i can be split into another pair of intervals (k, k') distinct from (j, j') then the length of k must be different from the length of j and the length of k' from the length of j'.

Although we have no precise definition of interval models yet, this property can be expressed formally for possible worlds models. The models satisfying this property are called S-models and the class of S-models is denoted by $\mathcal{C}$.

Definition 3.1 *A model $\mathcal{M} = (W, R, D, I)$ for a language $\mathcal{L}$ (which includes ℓ) is an S-model if for any world w, w_1, w_2, w_1', and w_2' of W such that $R(w_1, w_2, w)$ and $R(w_1', w_2', w)$,*

- *if $I(\ell, w_1) = I(\ell, w_1')$ then $w_2 = w_2'$ and*
- *if $I(\ell, w_2) = I(\ell, w_2')$ then $w_1 = w_1'$.*

The definition implies a *single decomposition* property: given three worlds w, w_1, w_2 such that $R(w_1, w_2, w)$, there is no world w_1' distinct from w_1 such that $R(w_1', w_2, w)$ and, symmetrically, there is no w_2' other than w_2 such that $R(w_1, w_2', w)$.

3.1.2 Proof system

We call S the deductive system which incorporates the following modal axioms:

A1:
$$(f;g) \wedge \neg(f;h) \Rightarrow (f; g \wedge \neg h)$$
$$(f;g) \wedge \neg(h;g) \Rightarrow (f \wedge \neg h; g)$$

R:
$$(f;g) \Rightarrow f \quad \text{if } f \text{ is a rigid formula}$$
$$(f;g) \Rightarrow g \quad \text{if } g \text{ is a rigid formula}$$

B:
$$((\exists x)f; g) \Rightarrow (\exists x)(f;g) \quad \text{if } x \text{ is not free in } g$$
$$(f; (\exists x)g) \Rightarrow (\exists x)(f;g) \quad \text{if } x \text{ is not free in } f$$

L1:
$$(\ell = x; f) \Rightarrow \neg(\ell = x; \neg f)$$
$$(f; \ell = x) \Rightarrow \neg(\neg f; \ell = x)$$

and the following inference rules

- modus ponens (MP): $\dfrac{f \quad f \Rightarrow g}{g}$,
- generalization (G): $\dfrac{f}{(\forall x) f}$,
- necessitation (N): $\dfrac{f}{\neg(\neg f; g)}$ and $\dfrac{f}{\neg(g; \neg f)}$,
- monotony (Mono): $\dfrac{f \Rightarrow g}{(f;h) \Rightarrow (g;h)}$ and $\dfrac{f \Rightarrow g}{(h;f) \Rightarrow (h;g)}$.

In addition, S contains first order and propositional axioms and axioms of identity for $\mathcal{L}$. The first order axioms can be chosen as in any axiomatic system for first order logic, except that some precaution must be taken in the instantiation of universally quantified formulas. For example, we can choose the two following quantification axioms:

Q1: $(\forall x)f(x) \Rightarrow f(t)$ if t is free for x in $f(x)$ and t is rigid or t is free for x in $f(x)$ and $f(x)$ is chop-free,

Q2: $(\forall x)(f \vee g) \Rightarrow ((\forall x)f) \vee g$ if x is not free in g.

The restrictions on Q1 prevent the substitution, in different modal contexts, of a (rigid) variable which represents a single object by a flexible term which may have different interpretations in different contexts.

As identity axioms, we can choose the axioms of reflexivity, symmetry, and transitivity of $=$, together with the following axioms for every functional symbol α and every predicate symbol ϕ (see [6, 13, 5]).

I1: $x_1 = y_1 \wedge \ldots \wedge x_n = y_n \Rightarrow \alpha(x_1, \ldots, x_n) = \alpha(y_1, \ldots, y_n)$
I2: $x_1 = y_x \wedge \ldots \wedge x_n = y_n \Rightarrow (\phi(x_1, \ldots, x_n) \Leftrightarrow \phi(y_1, \ldots, y_n))$.

where n is the arity of α or ϕ and $x_1, \ldots, x_n$, and $y_1, \ldots, y_n$ are arbitrary variables.

3.1.3 Soundness

The three pairs of axioms A1, R, and B are valid in ITL so they are also valid in $\mathcal{C}$. By the remarks of section 2.2.4 and the restriction on Q1, all the first order axioms are also valid. It is also easy to check that the definition of S-models ensures that L1 is valid in $\mathcal{C}$.

The four inference rules all preserve validity. Given a model $\mathcal{M}$, it is readily verified that any formula obtained by one of the rules MP, G, N, or Mono from formula(s) which are valid in $\mathcal{M}$ is also valid in $\mathcal{M}$.

It follows that the proof system is sound: any theorem of S is valid in $\mathcal{C}$.

3.2 Examples of theorems

In order to illustrate the use of the proof system, we give examples of theorems of S. Some of them have been proposed as possible axioms for the duration calculus or ITL [25, 21, 14] and others will be useful in the sequel for establishing completeness results.

3.2.1 Chop-Or

In section 2.2.4 we have stated that chop distributes over disjunctions. This can be derived in S. For example we show that $(f \vee g; h) \Leftrightarrow (f; h) \vee (g; h)$ is a theorem.

First, we derive the theorem $(f \vee g; h) \Rightarrow (f; h) \vee (g; h)$:

1	$(f \vee g; h) \wedge \neg(f; h) \Rightarrow ((f \vee g) \wedge \neg f; h)$	A1
2	$(f \vee g) \wedge \neg f \Rightarrow g$	Tauto
3	$((f \vee g) \wedge \neg f; h) \Rightarrow (g; h)$	Mono, 2
4	$(f \vee g; h) \wedge \neg(f; h) \Rightarrow (g; h)$	PC, 1, 3
5	$(f \vee g; h) \Rightarrow (f; h) \vee (g; h)$	PC, 4

The converse implication is also a theorem:

6	$f \Rightarrow f \vee g$	Tauto
7	$(f; h) \Rightarrow (f \vee g; h)$	Mono, 6
8	$g \Rightarrow f \vee g$	Tauto
9	$(g; h) \Rightarrow (f \vee g; h)$	Mono, 8
10	$(f; h) \vee (g; h) \Rightarrow (f \vee g; h)$	PC, 7, 9.

Then, the equivalence follows by propositional calculus. In the proof, PC and Tauto refer to elementary manipulations of predicate calculus: formula 2 is a tautology instance, formula 4 can be derived from 1 and 3 by MP and proposition calculus, etc.

Of course, the mirror of formula 5 is also a theorem. We call T1 any instance of the two following theorems:

$$\text{T1:} \quad \begin{array}{l} (f \vee g; h) \Rightarrow (f; h) \vee (g; h) \\ (f; g \vee h) \Rightarrow (f; g) \vee (f; h). \end{array}$$

In many existing proof systems, T1 is used as a fundamental axiom instead of A1 [25, 21, 26]. It is equivalent to replace A1 by T1 in S since A1 can be deduced from T1:

1	$f \Rightarrow (f \wedge \neg g) \vee g$	Tauto
2	$(f; h) \Rightarrow ((f \wedge \neg g) \vee g; h)$	Mono, 1
3	$((f \wedge \neg g) \vee g; h) \Rightarrow (f \wedge \neg g; h) \vee (g; h)$	T1
4	$(f; h) \Rightarrow (f \wedge \neg g; h) \vee (g; h)$	PC, 2, 3
5	$(f; h) \wedge \neg(g; h) \Rightarrow (f \wedge \neg g; h)$	PC, 4.

3.2.2 Quantification

A large number of proofs of first order logic can be carried out as well in S. In particular, variants of the quantification axioms Q1 and Q2 are useful theorems. If t is free for x in $f(x)$, and t is rigid or $f(x)$ chop-free, then the formula Q3 below is a theorem.

$$\text{Q3:} \quad f(t) \Rightarrow (\exists x) f(x).$$

If x is not free in g, then the three following formulas are theorems.

$$\begin{array}{ll} \text{Q4:} & (\exists x)(f \wedge g) \Rightarrow (\exists x) f \wedge g \\ \text{Q5:} & (\forall x)(f \Rightarrow g) \Rightarrow ((\exists x) f \Rightarrow g) \\ \text{Q6:} & (\forall x)(g \Rightarrow f) \Rightarrow (g \Rightarrow (\forall x) f). \end{array}$$

From these can be derived the reverse of Barcan's formula:

$$\text{T2:} \quad (\exists x)(f(x); g(x)) \Rightarrow ((\exists x) f(x); (\exists x) g(x)),$$

for example, as follows,

1	$f(x) \Rightarrow (\exists x) f(x)$	Q3
2	$g(x) \Rightarrow (\exists x) g(x)$	Q3
3	$(f(x); g(x)) \Rightarrow ((\exists x) f(x); (\exists x) g(x))$	Mono, 1,2
4	$(\forall x)((f(x); g(x)) \Rightarrow ((\exists x) f(x); (\exists x) g(x)))$	G, 3
5	$(\exists x)(f(x); g(x)) \Rightarrow ((\exists x) f(x); (\exists x) g(x))$	Q5, 4, MP.

3.2.3 Chop-Neg

Several useful theorems involve combinations of negations and chop, with conditions on length. A typical example is L1, from which follows immediately the two theorems:

$$(\ell = x; \neg f) \Rightarrow \neg(\ell = x; f)$$
$$(\neg f; \ell = x) \Rightarrow \neg(f; \ell = x).$$

Another useful theorem is the following:

$$\text{T3:} \quad \begin{array}{l} (\ell = x \wedge f; g) \Rightarrow \neg(\ell = x \wedge \neg f; h) \\ (f; \ell = x \wedge g) \Rightarrow \neg(h; \ell = x \wedge \neg g) \end{array}$$

where f, g, and h are arbitrary formulas.

These two formulas can be derived by introducing a variable y distinct from x and not occurring in f nor g. For example, for the first half of T3:

1	$g \Rightarrow (\ell = y) \vee \neg(\ell = y)$	Tauto
2	$(\ell = x \wedge f; g) \Rightarrow (\ell = x \wedge f; (\ell = y) \vee \neg(\ell = y))$	Mono, 1
3	$(\ell = x \wedge f; (\ell = y) \vee \neg(\ell = y)) \Rightarrow$ $(\ell = x \wedge f; \ell = y) \vee (\ell = x \wedge f; \neg(\ell = y))$	T1
4	$(\ell = x \wedge f; g) \Rightarrow (\ell = x \wedge f; \ell = y) \vee$ $(\ell = x \wedge f; \neg(\ell = y))$	PC, 2, 3

Both parts of the disjunction imply $\neg(\ell = x \wedge \neg f; \ell = y)$:

5	$(\ell = x \wedge f; \ell = y) \Rightarrow (f; \ell = y)$	PC, Mono
6	$(f; \ell = y) \Rightarrow \neg(\neg f; \ell = y)$	L1
7	$\neg(\neg f; \ell = y) \Rightarrow \neg(\ell = x \wedge \neg f; \ell = y)$	PC, Mono
8	$(\ell = x \wedge f; \ell = y) \Rightarrow \neg(\ell = x \wedge \neg f; \ell = y)$	PC, 5,6,7
9	$(\ell = x \wedge f; \neg(\ell = y)) \Rightarrow (\ell = x; \neg(\ell = y))$	PC, Mono
10	$(\ell = x; \neg(\ell = y)) \Rightarrow \neg(\ell = x; \neg\neg(\ell = y))$	L1
11	$\neg(\ell = x; \neg\neg(\ell = y)) \Rightarrow \neg(\ell = x \wedge \neg f; \ell = y)$	PC, Mono
12	$(\ell = x \wedge f; \neg(\ell = y)) \Rightarrow \neg(\ell = x \wedge \neg f; \ell = y)$	PC, 9,10,11

Then

13	$(\ell = x \wedge f; g) \Rightarrow \neg(\ell = x \wedge \neg f; \ell = y)$	PC, 4, 8, 12
14	$(\forall y)((\ell = x \wedge f; g) \Rightarrow \neg(\ell = x \wedge \neg f; \ell = y))$	G, 13
15	$(\ell = x \wedge f; g) \Rightarrow (\forall y)\neg(\ell = x \wedge \neg f; \ell = y)$	Q6, 14, MP
16	$(\ell = x \wedge f; g) \Rightarrow \neg(\exists y)(\ell = x \wedge \neg f; \ell = y)$	PC, 15.

On the other hand,

17	$h \Rightarrow (\exists y)(\ell = y)$	Ident, PC
18	$(\ell = x \wedge \neg f; h) \Rightarrow (\ell = x \wedge \neg f; (\exists y)(\ell = y))$	Mono, 17
19	$(\ell = x \wedge \neg f; (\exists y)(\ell = y)) \Rightarrow (\exists y)(\ell = x \wedge \neg f; \ell = y)$	B
20	$(\ell = x \wedge \neg f; h) \Rightarrow (\exists y)(\ell = x \wedge \neg f; \ell = y)$	PC, 18, 19

and finally:

21	$(\ell = x \wedge f; g) \Rightarrow \neg(\ell = x \wedge \neg f; h)$	PC, 20, 16.

3.2.4 Chop-And

Chop does not distribute over conjunctions in general. However, various restricted distributivity theorems can be derived for conjuncts of equal lengths. The simplest one may be

$$\text{T4:} \quad \begin{array}{l} (f; \ell = x) \wedge (g; \ell = x) \Rightarrow (f \wedge g; \ell = x) \\ (\ell = x; f) \wedge (\ell = x; g) \Rightarrow (\ell = x; f \wedge g). \end{array}$$

A possible proof is given below.

1	$(f; \ell = x) \Rightarrow \neg(\neg f; \ell = x)$	L1
2	$(g; \ell = x) \wedge \neg(\neg f; \ell = x) \Rightarrow (g \wedge \neg\neg f; \ell = x)$	A1
3	$g \wedge \neg\neg f \Rightarrow f \wedge g$	Tauto
4	$(g \wedge \neg\neg f; \ell = x) \Rightarrow (f \wedge g; \ell = x)$	Mono, 3
5	$(f; \ell = x) \wedge (g; \ell = x) \Rightarrow (f \wedge g; \ell = x)$	PC, 1–4.

By similar derivations, the following theorems can also be obtained:

$$\text{T5:} \quad \begin{array}{l} (f; g \wedge \ell = x) \wedge (h; \ell = x) \Rightarrow (f \wedge h; g \wedge \ell = x) \\ (f \wedge \ell = x; g) \wedge (\ell = x; h) \Rightarrow (f \wedge \ell = x; g \wedge h) \end{array}$$

$$\text{T6:} \quad \begin{array}{l} (f; g \wedge \ell = x) \wedge (h; g \wedge \ell = x) \Rightarrow (f \wedge h; g \wedge \ell = x) \\ (f \wedge \ell = x; g) \wedge (f \wedge \ell = x; h) \Rightarrow (f \wedge \ell = x; g \wedge h) \end{array}$$

$$\text{T7:} \quad (\ell = x; f) \wedge (g; \ell = y) \wedge (\ell = x; \ell = y) \Rightarrow (g \wedge \ell = x; f \wedge \ell = y).$$

For example, T5 can be proved as follows:

1	$(f; g \wedge \ell = x) \Rightarrow (f; \ell = x)$	Mono, PC
2	$(f; \ell = x) \wedge (h; \ell = x) \Rightarrow (f \wedge h; \ell = x)$	T4
3	$(f; g \wedge \ell = x) \Rightarrow \neg(f \wedge h; \neg g \wedge \ell = x)$	T3
4	$(f \wedge h; \ell = x) \wedge \neg(f \wedge h; \neg g \wedge \ell = x) \Rightarrow$ $(f \wedge h; \ell = x \wedge \neg(\neg g \wedge \ell = x))$	A1
5	$(f \wedge h; \ell = x \wedge \neg(\neg g \wedge \ell = x)) \Rightarrow$ $(f \wedge h; g \wedge \ell = x)$	Mono, PC
6	$(f; g \wedge \ell = x) \wedge (h; \ell = x) \Rightarrow (f \wedge h; g \wedge \ell = x)$	PC, 1–5.

T6 and T7 can be easily derived from T4 and T5.

Finally, the most general distributivity property of chop over conjunctions is given by the following theorem:

T8: $$\begin{array}{l}(f \wedge \ell = x; g) \wedge (h \wedge \ell = x; k) \Rightarrow (f \wedge h \wedge \ell = x; g \wedge k)\\ (f; g \wedge \ell = x) \wedge (h; k \wedge \ell = x) \Rightarrow (f \wedge h; g \wedge k \wedge \ell = x).\end{array}$$

This theorem can be derived as follows:

1	$(h \wedge \ell = x; k) \Rightarrow (\ell = x; k)$	PC, Mono
2	$(f \wedge \ell = x; g) \wedge (\ell = x; k) \Rightarrow (f \wedge \ell = x; g \wedge k)$	T5
3	$(h \wedge \ell = x; k) \Rightarrow \neg(\neg h \wedge \ell = x; g \wedge k)$	T3
4	$(f \wedge \ell = x; g \wedge k) \wedge \neg(\neg h \wedge \ell = x; g \wedge k) \Rightarrow$ $(f \wedge h \wedge \ell = x; g \wedge k)$	A1, PC, Mono
5	$(f \wedge \ell = x; g) \wedge (h \wedge \ell = x; k) \Rightarrow (f \wedge h \wedge \ell = x; g \wedge k)$	PC, 1–4.

3.3 Completeness

In this section, we show that S is complete: any formula f valid in $\mathcal{C}$ is provable in S. It is equivalent to show that any formula f such that $\neg f$ is not a theorem of S is satisfied in a model of $\mathcal{C}$. Our aim is to construct a model for any such formula.

It is sufficient to give a construction in the case where f is a closed formula; the general case will follow immediately. Also, instead of considering a single sentence f, it is simpler to generalize the construction to *consistent sets* of sentences, that is, roughly speaking, sets which do not contain contradictory sentences.

The essential result is that, for any consistent set Γ_0 in a language $\mathcal{L}$, we can construct an S-model $\mathcal{M}$ which satisfies Γ_0. The construction uses classic tools of first order and modal logic, namely maximal consistent sets and witnesses [11, 1, 18, 13, 6].

3.3.1 Consistent sets

For a formula f of an arbitrary ITL-language $\mathcal{L}$, $\vdash_S f$ and $\not\vdash_S f$ denote that f is or is not a theorem of S, respectively.

Given an arbitrary ITL-language $\mathcal{L}$, consistent and maximal consistent sets of sentences are defined in a standard way (for example, see chapter 9 in [18]):

Definition 3.2 *Let Γ be a set of sentences of $\mathcal{L}$,*

- Γ *is consistent (with respect to S) if there is no finite subset $\{f_1, \ldots, f_n\}$ of Γ such that*
$$\vdash_S \neg(f_1 \wedge \ldots \wedge f_n),$$
- Γ *is maximal consistent if it is consistent and there is no consistent set of sentences Γ' such that $\Gamma \subset \Gamma'$ (strictly).*

By propositional calculus, the following property is a straightforward consequence of the definition.

Proposition 3.3 *A consistent set of sentences Γ is maximal consistent if and only if, for every sentence f of $\mathcal{L}$ exactly one of f and $\neg f$ belongs to Γ.*

This implies that any maximal consistent set contains all the sentences which are theorems of S. The following properties are also easy consequences of the rules of propositional calculus.

Proposition 3.4 *Let Γ be a maximal consistent set and f and g two sentences of $\mathcal{L}$ then*

- $f \wedge g \in \Gamma$ *iff both* $f \in \Gamma$ *and* $g \in \Gamma$,
- $f \vee g \in \Gamma$ *iff* $f \in \Gamma$ *or* $g \in \Gamma$,
- *if* $f \Rightarrow g \in \Gamma$ *and* $f \in \Gamma$ *then* $g \in \Gamma$.

In ITL, maximal consistent sets have supplementary properties involving the chop operator:

Proposition 3.5 *Let Γ be a maximal consistent set and f, g, h, and k be four sentences of $\mathcal{L}$.*

- *If* $(f; g) \in \Gamma$, $\vdash_S f \Rightarrow h$, *and* $\vdash_S h \Rightarrow k$ *then* $(h; k) \in \Gamma$.
- *If* $(f; g) \in \Gamma$ *then* $\nvdash_S \neg f$ *and* $\nvdash_S \neg g$.

Proof: The first part follows from the monotony rule: if both $f \Rightarrow h$ and $g \Rightarrow k$ are theorems then, using Mono twice, $\vdash_S (f; g) \Rightarrow (h; k)$, so $(f; g) \Rightarrow (h; k)$ belongs to Γ. If in addition $(f; g) \in \Gamma$ then, by proposition 3.4, $(h; k) \in \Gamma$.

For the second part, assume one of $\neg f$ or $\neg g$ is a theorem, for example $\vdash_S \neg f$ then by rule N, $\neg(f; g)$ is a theorem and if $(f; g) \in \Gamma$, Γ is inconsistent. □

Finally, an essential property of consistent sets is given by Lidenbaum's lemma:

Theorem 3.6 (Lidenbaum) *For any consistent set Γ there is a maximal consistent set $\Gamma^\star$ such that $\Gamma \subseteq \Gamma^\star$.*

Proof: See [13] for example. □

3.3.2 Witnesses

In order to build a model for a consistent set Γ_0, we add a new set of constants to the language $\mathcal{L}$. These constants will serve as witnesses (see chapter 2 in [6]). More precisely, let $B = \{b_0, b_1, b_2, \ldots\}$ be an infinite, countable set of symbols not occurring in the language $\mathcal{L}$. We denote by $\mathcal{L}^+$ the ITL-language obtained by adding to $\mathcal{L}$ all the symbols of B as *rigid* individual constants. Hence, all the function and predicate symbols of $\mathcal{L}$ are also present in $\mathcal{L}^+$ with the same arity, rigid symbols of $\mathcal{L}$ are rigid in $\mathcal{L}^+$, and flexible symbols of $\mathcal{L}$ are flexible in $\mathcal{L}^+$.

With the expanded language $\mathcal{L}^+$ correspond new instances of the axioms of S. In particular, since all the constants $b_0, b_1, \ldots$ are rigid, $\mathcal{L}^+$ gives rise to new instances of the rigidity axiom R. We denote by S^+ the extended proof system and by $\vdash_{S^+}$ provability in S^+.

The model construction relies on the existence in $\mathcal{L}^+$ of sets of sentences which have the following property.

Definition 3.7 *A set Γ of sentences of $\mathcal{L}^+$ is said to have witnesses in B if for every sentence of Γ of the form $(\exists x)f(x)$ where x is the only free variable of $f(x)$ there exists a constant b_i of B such that $f(b_i)$ is also in Γ.*

This is a slight variation on the definition of [6]. The concept of witnesses for a set of sentences is also closely related to the notion of *omega-complete* sets used in [11] or [1].

The following theorem states a fundamental property of consistent sets.

Theorem 3.8 *If Γ is a consistent set of sentences of $\mathcal{L}$, there is a set $\Gamma^\star$ of sentences of $\mathcal{L}^+$ which satisfies the three following conditions:*

- $\Gamma \subseteq \Gamma^\star$,
- $\Gamma^\star$ *is maximal consistent,*
- $\Gamma^\star$ *has witnesses in B.*

Proof: The set $\Gamma^\star$ is obtained from Γ by the following standard construction (for example see [11] or [18] for modal logic, or [6, 5] for first order logic).

Since the language $\mathcal{L}^+$ contains countably many symbols, the set of sentences of $\mathcal{L}^+$ is countable. These sentences can then be enumerated in a sequence $f_0, f_1, f_2, \ldots$

We define a sequence of sets of sentences $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$ where $\Gamma_0 = \Gamma$ and then Γ_{i+1} is constructed from Γ_i as follows.

1. If $\Gamma_i \cup \{f_i\}$ is not consistent then
$$\Gamma_{i+1} \quad = \quad \Gamma_i \cup \{\neg f_i\},$$
2. If $\Gamma_i \cup \{f_i\}$ is consistent and f_i is of the form $(\exists x)g(x)$ then
$$\Gamma_{i+1} \quad = \quad \Gamma_i \cup \{f_i,\ g(b_j)\},$$
where b_j is a constant of B not occurring in any sentence of Γ_i.
3. If $\Gamma_i \cup \{f_i\}$ is consistent and f_i is not of the above form then
$$\Gamma_{i+1} \quad = \quad \Gamma_i \cup \{f_i\}.$$

In case 2, it is always possible to find an adequate constant b_j since only a finite number of constants of B can occur in Γ_i.

By induction, all the sets Γ_i can be shown to be consistent. This is true of $\Gamma_0 = \Gamma$ by assumption. By propositional calculus if Γ_i is consistent then $\Gamma_i \cup \{f_i\}$ and $\Gamma_i \cup \{\neg f_i\}$ cannot be both inconsistent, so in case 1 and (trivially) in case 3, Γ_{i+1} is consistent. In the remaining case, $\Gamma_i \cup \{f_i\} = \Gamma_i \cup \{(\exists x)g(x)\}$ is consistent. Assume Γ_{i+1} is not, then there are sentences $h_1, \ldots, h_n$ in Γ_i such that

$$\vdash_{S^+} \neg(h_1 \wedge \ldots \wedge h_n \wedge (\exists x)g(x) \wedge g(b_j)).$$

In a proof of this sentence, we can replace every occurrence of b_j by a variable y which is not already present, this yields a proof of the formula

$$\neg(h_1 \wedge \ldots \wedge h_n \wedge (\exists x)g(x) \wedge g(y)).$$

Then, by the generalization rule G,

$$\vdash_{S^+} (\forall y)\neg(h_1 \wedge \ldots \wedge h_n \wedge (\exists x)g(x) \wedge g(y)).$$

The term x is rigid and free for y in the above formula, so by $Q1$ and MP,

$$\vdash_{S^+} \neg(h_1 \wedge \ldots \wedge h_n \wedge (\exists x)g(x) \wedge g(x)),$$

using G again,

$$\vdash_{S^+} (\forall x)\neg(h_1 \wedge \ldots \wedge h_n \wedge (\exists x)g(x) \wedge g(x)),$$

and, by PC,

$$\vdash_{S^+} \neg(h_1 \wedge \ldots \wedge h_n \wedge (\exists x)g(x) \wedge (\exists x)g(x)),$$

$$\vdash_{S^+} \neg(h_1 \wedge \ldots \wedge h_n \wedge (\exists x)g(x)).$$

This contradicts the consistency of $\Gamma_i \cup \{f_i\}$ so Γ_{i+1} must be consistent.

Let $\Gamma^\star$ be the union of all the sets Γ_i, $\Gamma^\star$ is consistent since any finite subset of $\Gamma^\star$ is a subset of some Γ_i. It is also clear by construction that $\Gamma \subseteq \Gamma^\star$, that for any sentence $f = f_i$ of $\mathcal{L}^+$ either f_i or $\neg f_i$ belongs to $\Gamma^\star$ and that $\Gamma^\star$ has witnesses in B. Hence $\Gamma^\star$ satisfies the three conditions of the theorem. □

3.3.3 Model construction

By the preceding theorem, if Γ_0 is a consistent set of sentences of $\mathcal{L}$, there is a maximal consistent set $\Gamma_0^\star$ of sentences of $\mathcal{L}^+$ which has witnesses in B and such that $\Gamma_0 \subseteq \Gamma_0^\star$. We denote by Σ the set of rigid sentences of $\Gamma_0^\star$. We construct a model $\mathcal{M} = (W, R, D, I)$ where the worlds are sets of sentences of $\mathcal{L}^+$ which have certain desirable features and the domain is built from B and the set Σ.

Frame

We introduce the following notation: given two sets of sentences Γ_1 and Γ_2, $\Gamma_1 * \Gamma_2$ denotes the set of sentences $(f_1; f_2)$ with f_1 in Γ_1 and f_2 in Γ_2. Then the frame (W, R) is defined as follows.

- ⋄ The set of worlds W is the set of all maximal consistent sets Δ of $\mathcal{L}^+$ which have witnesses in B and such that $\Sigma \subseteq \Delta$.
- ⋄ The relation R is defined by

$$R(\Delta_1, \Delta_2, \Delta) \quad \text{iff} \quad \Delta_1 * \Delta_2 \subseteq \Delta,$$

for all Δ_1, Δ_2 and Δ of W. In other words, a world Δ of W can be decomposed into a pair of worlds (Δ_1, Δ_2) if and only if for any f_1 of Δ_1 and f_2 of Δ_2 the sentence $(f_1; f_2)$ is in Δ.

By construction, it is easy to see that the rigid sentences of any set Δ of W are exactly the elements of Σ. To show this, assume Δ contains a rigid sentence f which is not in Σ. Then f is not in $\Gamma_0^\star$ either and, since $\Gamma_0^\star$ is maximal consistent, $\neg f$ is in $\Gamma_0^\star$. But $\neg f$ is a rigid sentence and belongs to Σ. Since $\Sigma \subseteq \Delta$ this contradicts the consistency of Δ.

Domain

On the set B we define a binary relation $\equiv$ as follows: for b_i and b_j of B,

$$b_i \equiv b_j \quad \text{iff} \quad (b_i = b_j) \in \Sigma.$$

By the axioms of identity, $\equiv$ is an equivalence relation on B. For example, to show that $\equiv$ is transitive, assume $b_i \equiv b_j$ and $b_j \equiv b_k$. By definition, $(b_i = b_j)$ and $(b_j = b_k)$ are two sentences of Σ and then of $\Gamma_0^\star$. By the axioms of identity,

$$\vdash_{S+} (b_i = b_j) \wedge (b_j = b_k) \Rightarrow (b_i = b_k).$$

Since $\Gamma_0^\star$ is maximal consistent it follows by proposition 3.4 that $(b_i = b_k)$ belongs to $\Gamma_0^\star$. This sentence is rigid, so $(b_i = b_k) \in \Sigma$, that is, $b_i \equiv b_k$. Symmetry an reflexivity can be proved in a similar way (see [6]).

For any constant b_i of B, we denote by $[b_i]$ the equivalence class of b_i and we define the domain D of $\mathcal{M}$ by:

$$D \quad = \quad \{\, [b_i] \mid b_i \in B \,\}.$$

The domain of $\mathcal{M}$ is then the set of the equivalence classes of $\equiv$.

Interpretation function

It remains to define the interpretation function I. In an arbitrary world Δ, the interpretation of a symbol of $\mathcal{L}^+$ is defined as in [6], chapter 2.

For a proposition symbol p, we simply set

$$I(p, \Delta) = 1 \quad \text{iff} \quad p \in \Delta.$$

For an n-ary predicate symbol ϕ, let $b_{i_1}, \ldots, b_{i_n}$ and $b_{i'_1}, \ldots, b_{i'_n}$ be constants of B. By the axioms of identity,

$$\vdash_{S^+} (b_{i_1} = b_{i'_1}) \wedge \ldots \wedge (b_{i_n} = b_{i'_n}) \Rightarrow (\phi(b_{i_1}, \ldots, b_{i_n}) \Leftrightarrow \phi(b_{i'_1}, \ldots, b_{i'_n})).$$

If $[b_{i_1}] = [b_{i'_1}], \ldots, [b_{i_n}] = [b_{i'_n}]$, all the sentences $(b_{i_1} = b_{i'_1}), \ldots, (b_{i_n} = b_{i'_n})$ are in Σ. Since $\Sigma \subseteq \Delta$, they are also in Δ and since Δ is maximal consistent,

$$\phi(b_{i_1}, \ldots, b_{i_n}) \Leftrightarrow \phi(b_{i'_1}, \ldots, b_{i'_n})$$

is a sentence of Δ. Then, by proposition 3.4,

$$\phi(b_{i_1}, \ldots, b_{i_n}) \in \Delta \quad \text{iff} \quad \phi(b_{i'_1}, \ldots, b_{i'_n}) \in \Delta.$$

This equivalence makes it possible to define $I(\phi, \Delta)$ as the n-ary relation on D such that,

$$I(\phi, \Delta)([b_{i_1}], \ldots, [b_{i_n}]) = 1 \quad \text{iff} \quad \phi(b_{i_1}, \ldots, b_{i_n}) \in \Delta$$

for any constants $b_{i_1}, \ldots, b_{i_n}$ of B.

For an individual constant a, by the axioms of identity and predicate calculus, we have

$$\vdash_{S^+} (\exists x)(a = x).$$

The sentence $(\exists x)(a = x)$ is then in Δ and, since Δ has witnesses in B, there is a constant b_j of B such that $(a = b_j)$ is in Δ. The interpretation of a in Δ is defined by $I(a, \Delta) = [b_j]$. This is independent of a particular choice of b_j for, if $b_{j'}$ is another constant of B, we have

$$\vdash_{S^+} (a = b_j) \wedge (a = b_{j'}) \Rightarrow (b_j = b'_j).$$

Hence, for any constant b_j of B,

$$I(a, \Delta) = [b_j] \quad \text{iff} \quad (a = b_j) \in \Delta.$$

For an n-ary function symbol α, let $b_{i_1}, \ldots, b_{i_n}$ be n constants of B. By the axioms of identity,

$$\vdash_{S^+} (\exists x)(\alpha(b_{i_1}, \ldots, b_{i_n}) = x)$$

and, as previously, there is a constant b_j such that $\alpha(b_{i_1}, \ldots, b_{i_n}) = b_j$ belongs to Δ. We set

$$I(\alpha, \Delta)([b_{i_1}], \ldots, [b_{i_n}]) = [b_j]$$

and this is again independent of the choice of class representatives. For any constant $b_{i_1}, \ldots, b_{i_n}$ and b_j of B, the definition ensures that

$$I(\alpha, \Delta)([b_{i_1}], \ldots, [b_{i_n}]) = [b_j] \quad \text{iff} \quad (\alpha(b_{i_1}, \ldots, b_{i_n}) = b_j) \in \Delta.$$

Since the rigid sentences of all the worlds Δ of W are the same, the function I is correctly defined: all the rigid symbols have the same interpretation in all the worlds.

3.3.4 Completeness theorem

The preceding construction yields a model $\mathcal{M}$ from any consistent set Γ_0 of sentences of $\mathcal{L}$. We have to verify that $\mathcal{M}$ is a model of Γ_0 and that $\mathcal{M}$ is an S-model.

By construction, a proposition p of $\mathcal{L}^+$ is satisfied in a world Δ of $\mathcal{M}$ if and only if p belongs to Δ. This also holds for atomic formulas of the form $\phi(b_{i_1}, \ldots, b_{i_n})$. The proof that $\mathcal{M}$ satisfies Γ_0 relies on a generalization of the latter property: an arbitrary sentence f is satisfied in a world Δ if and only if f belongs to Δ. This is shown by classic means (see [18, 11]) the only difficulty is the case of chop formulas.

The main lemmas

The main step is to show that, if a chop formula $(f_1; f_2)$ belongs to a world Δ of $\mathcal{M}$, there are two worlds Δ_1 and Δ_2 such that $f_1 \in \Delta_1$, $f_2 \in \Delta_2$, and $\Delta_1 * \Delta_2 \subseteq \Delta$. In order to establish this property, we will use the following notations. Given a non-empty set of sentences Γ, we denote by $\widehat{\Gamma}$ and $\overline{\Gamma}$ the two sets:

$$\begin{aligned} \widehat{\Gamma} &= \{h_1 \wedge \ldots \wedge h_m \mid m \geqslant 1, h_1 \in \Gamma, \ldots, h_m \in \Gamma\}, \\ \overline{\Gamma} &= \{h \mid \vdash_{S^+} (f \Rightarrow h) \quad \text{for some} \quad f \in \widehat{\Gamma}\}. \end{aligned}$$

$\widehat{\Gamma}$ is the set of conjunctions of sentences of Γ and $\overline{\Gamma}$ the set of consequences of sentences of Γ. We always have $\Gamma \subseteq \widehat{\Gamma} \subseteq \overline{\Gamma}$ and Γ is consistent if and only if $\overline{\Gamma}$ is not the set of all sentences of $\mathcal{L}^+$. If Γ is maximal consistent then $\Gamma = \widehat{\Gamma} = \overline{\Gamma}$.

Let Γ be a maximal consistent set and Γ_1 and Γ_2 be two non-empty sets of sentences. We will show that if $\widehat{\Gamma_1} * \widehat{\Gamma_2} \subseteq \Gamma$ there are two maximal consistent sets $\Gamma_1^\star$ and $\Gamma_2^\star$ such that $\Gamma_1 \subseteq \Gamma_1^\star$, $\Gamma_2 \subseteq \Gamma_2^\star$ and $\Gamma_1^\star * \Gamma_2^\star \subseteq \Gamma$. The idea is to construct from Γ_1 and Γ_2 two maximal consistent sets $\Gamma_1^\star$ and $\Gamma_2^\star$ in such a way that for any sentence $\neg(f_1; f_2)$ of Γ, $\neg f_1$ is in $\Gamma_1^\star$ or $\neg f_2$ is in $\Gamma_2^\star$. The construction relies on the two following lemmas.

Lemma 3.9 *If Γ_1 and Γ_2 are non-empty and $\widehat{\Gamma_1} * \widehat{\Gamma_2} \subseteq \Gamma$ then Γ_1 and Γ_2 are consistent and $\overline{\Gamma_1} * \overline{\Gamma_2} \subseteq \Gamma$.*

Proof: Assume one of Γ_1 or Γ_2 is inconsistent, say Γ_1, then there are sentences $f_1, \ldots, f_n$ of Γ_1 such that

$$\vdash_{S^+} \neg(f_1 \wedge \ldots \wedge f_n).$$

Let g be a sentence of Γ_2; by the necessity rule N,

$$\vdash_{S^+} \neg(f_1 \wedge \ldots \wedge f_n; g).$$

Since Γ is consistent, the sentence $(f_1 \wedge \ldots \wedge f_n; g)$ cannot be in Γ and this contradicts the assumption that $\widehat{\Gamma_1} * \widehat{\Gamma_2} \subseteq \Gamma$.

For the second part of the lemma, let f and g be two sentences of $\overline{\Gamma_1}$ and $\overline{\Gamma_2}$, respectively. By definition, there are $f_1, \ldots, f_n$ in Γ_1 and $g_1, \ldots, g_m$ in Γ_2 such that

$$\vdash_{S+} (f_1 \wedge \ldots \wedge f_n) \Rightarrow f \quad \text{and} \quad \vdash_{S+} (g_1 \wedge \ldots \wedge g_m) \Rightarrow g.$$

Using Mono twice,

$$\vdash_{S+} (f_1 \wedge \ldots \wedge f_n; g_1 \wedge \ldots \wedge g_m) \Rightarrow (f; g).$$

By assumption, $(f_1 \wedge \ldots \wedge f_n; g_1 \wedge \ldots \wedge g_m)$ belongs to Γ therefore $(f; g)$ is also a sentence of Γ. □

The second lemma uses the two following functions defined for arbitrary sets of sentences Γ, Γ_1, and Γ_2:

$$\begin{aligned} \delta_1(\Gamma, \Gamma_1) &= \{\neg g | \neg(f; g) \in \Gamma, f \in \Gamma_1\}, \\ \delta_2(\Gamma, \Gamma_2) &= \{\neg f | \neg(f; g) \in \Gamma, g \in \Gamma_2\}. \end{aligned}$$

Lemma 3.10 *Given a maximal consistent set Γ and two non-empty sets Γ_1 and Γ_2 such that $\widehat{\Gamma_1} * \widehat{\Gamma_2} \subseteq \Gamma$, let Γ_1' and Γ_2' be defined as follows:*

$$\Gamma_1' = \Gamma_1 \cup \delta_2(\Gamma, \Gamma_2) \quad \text{and} \quad \Gamma_2' = \Gamma_2 \cup \delta_1(\Gamma, \Gamma_1),$$

then

$$\Gamma_1' * \Gamma_2 \subseteq \Gamma \quad \text{and} \quad \Gamma_1 * \Gamma_2' \subseteq \Gamma.$$

Proof: The two cases are symmetrical, we show the inclusion for Γ_1'.

Let $f_1', \ldots, f_n'$ be n sentences of Γ_1' and $g_1, \ldots, g_l$ be l sentences of Γ_2. If all the formulas $f_1', \ldots, f_n'$ are in Γ_1 then $(f_1' \wedge \ldots \wedge f_n'; g_1 \wedge \ldots \wedge g_l)$ is in Γ by assumption.

Otherwise, some of the sentences $f_1', \ldots, f_n'$ come from $\delta_2(\Gamma, \Gamma_2)$. Without loss of generality, we can assume that these sentences are $f_1', \ldots, f_m'$ for some $m \leqslant n$.

By definition of δ_2, there are formulas $f_1, \ldots, f_m$ and $h_1, \ldots, h_m$ such that, for $i = 1, \ldots, m$,

- ◇ f_i' is the sentence $\neg f_i$,
- ◇ h_i belongs to Γ_2,
- ◇ $\neg(f_i; h_i)$ belongs to Γ.

Let g be the conjunction $g_1 \wedge \ldots \wedge g_l \wedge h_1 \wedge \ldots \wedge h_m$. We can derive

1	$g \Rightarrow h_i$	Tauto
2	$(f_i; g) \Rightarrow (f_i; h_i)$	Mono, 1
3	$\neg(f_i; h_i) \Rightarrow \neg(f_i; g)$	PC, 2,

thus, since Γ is maximal consistent, all the sentences $\neg(f_i; g)$ are in Γ.

If $m < n$, let f be the sentence $f'_{m+1} \wedge \ldots \wedge f'_n$ else let f be an arbitrary sentence of Γ_2. g is a conjunction of sentences of Γ_2 and f a conjunction of sentences of Γ_1 therefore $(f; g)$ is in Γ.

The following theorem

$$\vdash_{S^+} (f; g) \wedge \neg(f_1; g) \wedge \ldots \wedge \neg(f_m; g) \Rightarrow (f \wedge \neg f_1 \wedge \ldots \wedge \neg f_m; g)$$

can be derived using A1 repeatedly. It follows that the sentence

$$(f \wedge \neg f_1 \wedge \ldots \wedge \neg f_m; g)$$

belongs to Γ. By construction, we have

$$\vdash_{S^+} f \wedge \neg f_1 \ldots \wedge \neg f_m \Rightarrow f'_1 \wedge \ldots \wedge f'_n \quad \text{and} \quad \vdash_{S^+} g \Rightarrow g_1 \wedge \ldots \wedge g_l,$$

so, by proposition 3.5, $(f'_1 \wedge \ldots \wedge f'_n; g_1 \wedge \ldots \wedge g_l)$ is in Γ. □

We can now show the essential result, stated by the following theorem.

Theorem 3.11 *If Γ is maximal consistent and Γ_1 and Γ_2 are two non-empty sets of sentences such that $\widehat{\Gamma_1} * \widehat{\Gamma_2} \subseteq \Gamma$ then there are two maximal consistent sets $\Gamma_1^\star$ and $\Gamma_2^\star$ such that*

- ◇ $\Gamma_1 \subseteq \Gamma_1^\star$,
- ◇ $\Gamma_2 \subseteq \Gamma_2^\star$,
- ◇ $\Gamma_1^\star * \Gamma_2^\star \subseteq \Gamma$.

Proof: We construct recursively two sequences $\Gamma_1^{(n)}$ and $\Gamma_2^{(n)}$ of sets of sentences. $\Gamma_1^{(0)}$ and $\Gamma_2^{(0)}$ are defined by

$$\Gamma_1^{(0)} = \overline{\Gamma_1} \quad \text{and} \quad \Gamma_2^{(0)} = \overline{\Gamma_2}$$

and $\Gamma_1^{(n+1)}$ and $\Gamma_2^{(n+1)}$ are obtained from $\Gamma_1^{(n)}$ and $\Gamma_2^{(n)}$ as follows:

- ◇ for n even,

$$\Gamma_1^{(n+1)} = \overline{\Gamma_1^{(n)} \cup \delta_2(\Gamma, \Gamma_2^{(n)})} \quad \text{and} \quad \Gamma_2^{(n+1)} = \Gamma_2^{(n)},$$

- ◇ for n odd,

$$\Gamma_1^{(n+1)} = \Gamma_1^{(n)} \quad \text{and} \quad \Gamma_2^{(n+1)} = \overline{\Gamma_2^{(n)} \cup \delta_1(\Gamma, \Gamma_1^{(n)})}.$$

By assumption, $\widehat{\Gamma_1} * \widehat{\Gamma_2} \subseteq \Gamma$, so, by lemma 3.9, $\Gamma_1^{(0)} * \Gamma_2^{(0)} \subseteq \Gamma$. By induction and lemma 3.10 we have, for all n,

$$\Gamma_1^{(n)} * \Gamma_2^{(n)} \subseteq \Gamma.$$

Let Γ_1^ω and Γ_2^ω be the unions of the sets $\Gamma_1^{(n)}$ and $\Gamma_2^{(n)}$, respectively. If $f_1, \ldots, f_m$ are in Γ_1^ω and $g_1, \ldots, g_l$ in Γ_2^ω then there is an index n such that

$$\{f_1, \ldots, f_m\} \subseteq \Gamma_1^{(n)} \quad \text{and} \quad \{g_1, \ldots, g_l\} \subseteq \Gamma_2^{(n)}.$$

It follows that $\Gamma_1^\omega * \Gamma_2^\omega \subseteq \Gamma$ and, by lemma 3.9, both Γ_1^ω and Γ_2^ω are consistent. By Lidenbaum's lemma (theorem 3.6), there exists a maximal consistent set $\Gamma_1^\star$ such that $\Gamma_1^\omega \subseteq \Gamma_1^\star$.

Consider a sentence g of Γ_2^ω and an arbitrary sentence f such that $\neg(f; g)$ is in Γ. There is an index n such that $g \in \Gamma_2^{(n)}$ and then $\neg f \in \delta_2(\Gamma, \Gamma_2^{(n)})$. This clearly implies that $\neg f$ is in Γ_1^ω and also in $\Gamma_1^\star$. Hence for any sentence f of $\Gamma_1^\star$ and any g of Γ_2^ω we have $(f; g) \in \Gamma$, that is,

$$\Gamma_1^\star * \Gamma_2^\omega \subseteq \Gamma.$$

Since $\Gamma_1^\star$ is maximal consistent, $\widehat{\Gamma_1^\star} = \Gamma_1^\star$. By construction, $\Gamma_2^\omega = \widehat{\Gamma_2^\omega} = \overline{\Gamma_2^\omega}$, thus

$$\widehat{\Gamma_1^\star} * \widehat{\Gamma_2^\omega} \subseteq \Gamma.$$

Let Γ_2' be the set $\Gamma_2^\omega \cup \delta_1(\Gamma, \Gamma_1^\star)$. By lemma 3.10,

$$\Gamma_1^\star * \Gamma_2' \subseteq \Gamma$$

and, by lemma 3.9, Γ_2' is consistent. By Lidenbaum's lemma, there is a maximal consistent extension $\Gamma_2^\star$ of Γ_2'. As previously, if f is in $\Gamma_1^\star$ and g is a sentence such that $\neg(f; g)$ belongs to Γ then $\neg g$ is in $\delta_1(\Gamma, \Gamma_1^\star)$ and also in Γ_2' and $\Gamma_2^\star$. For any sentence f of $\Gamma_1^\star$ and g of $\Gamma_2^\star$ the sentence $(f; g)$ is then in Γ, hence

$$\Gamma_1^\star * \Gamma_2^\star \subseteq \Gamma.$$

By construction, it is clear that $\Gamma_1 \subseteq \Gamma_1^\star$ and $\Gamma_2 \subseteq \Gamma_2^\star$; $\Gamma_1^\star$ and $\Gamma_2^\star$ satisfy the three required conditions. □

Finally, the following lemma gives a sufficient condition for two maximal consistent sets $\Gamma_1^\star$ and $\Gamma_2^\star$ to be worlds of $\mathcal{M}$.

Lemma 3.12 *Let Δ be a world of $\mathcal{M}$ and $\Gamma_1^\star$ and $\Gamma_2^\star$ be two maximal consistent sets of sentences of $\mathcal{L}^+$. If the following three conditions are satisfied:*

- ◇ $\Gamma_1^\star * \Gamma_2^\star \subseteq \Delta$,
- ◇ *there is an element b_i of B such that $(\ell = b_i)$ is a sentence of $\Gamma_1^\star$,*
- ◇ *there is an element b_j of B such that $(\ell = b_j)$ is a sentence of $\Gamma_2^\star$*

then $\Gamma_1^\star$ and $\Gamma_2^\star$ are two worlds of $\mathcal{M}$.

Proof: We have to show that Σ is included in $\Gamma_1^\star$ and $\Gamma_2^\star$ and that the two sets have witnesses in B.

Let f be a sentence of Σ. f is a rigid sentence and its negation is also rigid. By axiom R,

$$\vdash_{S+} (\neg f; \ell = b_j) \Rightarrow \neg f.$$

Assume f does not belong to $\Gamma_1^\star$ then $\neg f$ is in $\Gamma_1^\star$ and, since $\Gamma_1^\star * \Gamma_2^\star \subseteq \Delta$,

$$(\neg f; \ell = b_j) \in \Delta$$

then $\neg f$ is in Δ too. But this contradicts the assumption that Δ is maximal consistent and contains Σ. Hence, every sentence of Σ must be in $\Gamma_1^\star$ and, symmetrically, in $\Gamma_2^\star$.

Let $(\exists x)f(x)$ be a sentence of $\Gamma_1^\star$ then $((\exists x)f(x); \ell = b_j) \in \Delta$.
The formula $\ell = b_j$ does not contain x so by Barcan's formula,

$$\vdash_{S+} ((\exists x)f(x); \ell = b_j) \Rightarrow (\exists x)(f(x); \ell = b_j).$$

Then $(\exists x)(f(x); \ell = b_j)$ is in Δ and since Δ has witnesses in B there is a constant b_k such that

$$(f(b_k); \ell = b_j) \in \Delta.$$

By L1,

$$\vdash_{S+} (f(b_k); \ell = b_j) \Rightarrow \neg(\neg f(b_k); \ell = b_j),$$

therefore $\neg(\neg f(b_k); \ell = b_j)$ is a sentence of Δ. As a consequence, $\neg f(b_k)$ cannot be in $\Gamma_1^\star$ and $f(b_k)$ belongs to $\Gamma_1^\star$. Hence $\Gamma_1^\star$ has witnesses in B. A symmetrical proof shows that $\Gamma_2^\star$ also has witnesses in B. □

$\mathcal{M}$ satisfies Γ_0

The following two theorems state properties of $\mathcal{M}$ which will ensure that $\mathcal{M}$ is actually a model of Γ_0.

Theorem 3.13 *Let $t(x_1, \ldots, x_n)$ be a term with variables among $x_1, \ldots, x_n$. Let $b_{i_1}, \ldots, b_{i_n}$ be n constants of B and v be an $\mathcal{M}$-valuation such that*

$$v(x_1) = [b_{i_1}], \ldots, v(x_n) = [b_{i_n}],$$

then for any b_j of B and any world Δ of W,

$$I^v_\Delta(t(x_1, \ldots, x_n)) = [b_j] \quad \text{iff} \quad (t(b_{i_1}, \ldots, b_{i_n}) = b_j) \in \Delta.$$

Proof: The proof is by induction on terms.

- ⋄ If $t(x_1, \ldots, x_n)$ is an individual constant a then

$$I^v_\Delta(t(x_1, \ldots, x_n)) = I(a, \Delta)$$

and by construction of the interpretation function, $I(a, \Delta) = [b_j]$ if and only if $(a = b_j)$ is a sentence of Δ.

- ⋄ If $t(x_1, \ldots, x_n)$ is a variable x_k $(1 \leqslant k \leqslant n)$ then

$$I^v_\Delta(t(x_1, \ldots, x_n) = v(x_k) = [b_{i_k}].$$

By definition of $\equiv$ we have

$$[b_{i_k}] = [b_j] \quad \text{iff} \quad (b_{i_k} = b_j) \in \Sigma$$

and by construction of $\mathcal{M}$, this is equivalent to $(b_{i_k} = b_j) \in \Delta$.

- If $t(x_1, \ldots, x_n)$ is of the form $\alpha(t_1(x_1, \ldots, x_n), \ldots, t_m(x_1, \ldots, x_n))$ for a function symbol α of arity m then

  $$I^v_\Delta(t(x_1, \ldots, x_n)) = I(\alpha, \Delta)(I^v_\Delta(t_1(x_1, \ldots, x_n)), \ldots, I^v_\Delta(t_m(x_1, \ldots, x_n)))$$

  and the equivalence follows by induction and definition of $I(\alpha, \Delta)$ (see [6]).

□

Theorem 3.14 *Let $f(x_1, \ldots, x_n)$ be a formula of $\mathcal{L}^+$ with free variables among $x_1, \ldots, x_n$. For any world Δ of W, any $\mathcal{M}$-valuation v and any constants $b_{i_1}, \ldots, b_{i_n}$ such that*

$$v(x_1) = [b_{i_1}], \ldots, v(x_n) = [b_{i_n}],$$

we have,

$$\mathcal{M}, \Delta, v \models f(x_1, \ldots, x_n) \quad \text{iff} \quad f(b_{i_1}, \ldots, b_{i_n}) \in \Delta.$$

Proof: The proof is by induction on $f(x_1, \ldots, x_n)$.

- For atomic formulas, the equivalence follows from the definition of the interpretation function I and theorem 3.13.

- If $f(x_1, \ldots, x_n)$ is of the form $f_1 \wedge f_2$ or $\neg f_1$ then the equivalence is shown by induction and the properties of maximal consistent sets (see [13] for example).

- For formulas $f(x_1, \ldots, x_n)$ of the form $(\exists x_{n+1})g(x_1, \ldots, x_{n+1})$ the result is true because every world Δ has witnesses in B:

  If $\mathcal{M}, \Delta, v \models f(x_1, \ldots, x_n)$, there is a valuation v' such that

  $$v(x_1) = v'(x_1), \ldots, v(x_n) = v'(x_n), \quad \text{and} \quad \mathcal{M}, \Delta, v' \models g(x_1, \ldots, x_{n+1}).$$

  Let $b_{i_1}, \ldots, b_{i_{n+1}}$ be elements of B such that

  $$v'(x_1) = [b_{i_1}], \ldots, v'(x_{n+1}) = [b_{i_{n+1}}],$$

  then by the induction hypothesis,

  $$g(b_{i_1}, \ldots, b_{i_{n+1}}) \in \Delta.$$

  Since $b_{i_{n+1}}$ is rigid, Q3 yields

  $$\vdash_{S^+} g(b_{i_1}, \ldots, b_{i_{n+1}}) \Rightarrow (\exists x_{n+1})g(b_{i_1}, \ldots, b_{i_n}, x_{n+1})$$

  then, since Δ is maximal consistent,

  $$(\exists x_{n+1})g(b_{i_1}, \ldots, b_{i_n}, x_{n+1}) \in \Delta,$$

  that is $f(b_{i_1}, \ldots, b_{i_n}) \in \Delta$.

Conversely, let $b_{i_1}, \ldots, b_{i_n}$ be n constants of B such that

$$v(x_1) = [b_{i_1}], \ldots, v(x_n) = [b_{i_n}]$$

and assume the sentence $(\exists x_{n+1})g(b_{i_1}, \ldots, b_{i_n}, x_{n+1})$ is in Δ. Since Δ has witnesses in B there is a constant $b_{i_{n+1}}$ such that

$$g(b_{i_1}, \ldots, b_{i_n}, b_{i_{n+1}}) \in \Delta.$$

Let v' be an $\mathcal{M}$-valuation such that

$$v'(x_1) = [b_{i_1}], \ldots, v'(x_{n+1}) = [b_{i_{n+1}}],$$

then by the induction hypothesis,

$$\mathcal{M}, \Delta, v' \models g(x_1, \ldots, x_{n+1})$$

and

$$\mathcal{M}, \Delta, v \models (\exists x_{n+1})g(x_1, \ldots, x_{n+1}).$$

◇ For chop formulas $(g(x_1, \ldots, x_n); h(x_1, \ldots, x_n))$ the proof relies on theorem 3.11. Let $b_{i_1}, \ldots, b_{i_n}$ be n constants such that

$$v(x_1) = [b_{i_1}], \ldots, v(x_n) = [b_{i_n}].$$

If $\mathcal{M}, \Delta, v \models (g(x_1, \ldots, x_n); h(x_1, \ldots, x_n))$, there are two worlds Δ_1 and Δ_2 such that

$$\begin{array}{rcl} \mathcal{M}, \Delta_1, v & \models & g(x_1, \ldots, x_n), \\ \mathcal{M}, \Delta_2, v & \models & h(x_1, \ldots, x_n), \\ \Delta_1 * \Delta_2 & \subseteq & \Delta. \end{array}$$

By the induction hypothesis, this implies that

$$g(b_{i_1}, \ldots, b_{i_n}) \in \Delta_1 \quad \text{and} \quad h(b_{i_1}, \ldots, b_{i_n}) \in \Delta_2$$

and, since $\Delta_1 * \Delta_2 \subseteq \Delta$,

$$(g(b_{i_1}, \ldots, b_{i_n}); h(b_{i_1}, \ldots, b_{i_n})) \in \Delta.$$

Conversely, assume

$$(g(b_{i_1}, \ldots, b_{i_n}); h(b_{i_1}, \ldots, b_{i_n})) \in \Delta.$$

Let g' and h' denote the sentences $g(b_{i_1}, \ldots, b_{i_n})$ and $h(b_{i_1}, \ldots, b_{i_n})$, respectively, and let x and y be two variables, we can derive

1	$g' \Rightarrow (\exists x)(g' \wedge \ell = x)$	PC
2	$h' \Rightarrow (\exists y)(h' \wedge \ell = y)$	PC
3	$(g'; h') \Rightarrow (\ (\exists x)(g' \wedge \ell = x)\ ;\ (\exists y)(h' \wedge \ell = y)\ )$	Mono, 1, 2
4	$((\exists x)(g' \wedge \ell = x); (\exists y)(h' \wedge \ell = y)) \Rightarrow$ $(\exists x)(\exists y)(g' \wedge \ell = x; h' \wedge \ell = y)$	B
5	$(g'; h') \Rightarrow (\exists x)(\exists y)(g' \wedge \ell = x; h' \wedge \ell = y)$	PC, 3, 4

Then the sentence $(\exists x)(\exists y)(g' \wedge \ell = x; h' \wedge \ell = y)$ belongs to Δ. Since Δ has witnesses in B there are two constants b_i and b_j such that

$$(g' \wedge \ell = b_i; h' \wedge \ell = b_j) \in \Delta.$$

Let Γ_1 and Γ_2 be the two following sets of sentences:

$$\begin{aligned} \Gamma_1 &= \{g', \ell = b_i\} \\ \Gamma_2 &= \{h', \ell = b_j\}. \end{aligned}$$

It is clear that $\widehat{\Gamma_1} * \widehat{\Gamma_2} \subseteq \Delta$, we can then apply theorem 3.11: there are two maximal consistent sets $\Gamma_1^\star$ and $\Gamma_2^\star$ such that

$$\Gamma_1 \subseteq \Gamma_1^\star, \quad \Gamma_2 \subseteq \Gamma_2^\star, \quad \text{and} \quad \Gamma_1^\star * \Gamma_2^\star \subseteq \Delta.$$

By lemma 3.12 the two sets $\Gamma_1^\star$ and $\Gamma_2^\star$ are worlds of $\mathcal{M}$. By induction, since $g' \in \Gamma_1^\star$ and $h' \in \Gamma_2^\star$,

$$\mathcal{M}, \Gamma_1^\star, v \models g(x_1, \ldots, x_n) \quad \text{and} \quad \mathcal{M}, \Gamma_2^\star, v \models h(x_1, \ldots, x_n),$$

and then

$$\mathcal{M}, \Delta, v \models (g(x_1, \ldots, x_n); h(x_1, \ldots, x_n)).$$

□

Corollary 3.15 *$\mathcal{M}$ is a model of Γ_0.*

Proof: By construction, there is a set Δ_0 of $\mathcal{M}$ such that $\Gamma_0 \subseteq \Delta_0$. If f is a sentence of Γ_0, theorem 3.14 shows that f is satisfied in Δ_0 under any valuation v. □

$\mathcal{M}$ is an S-model

The following proposition states that $\mathcal{M}$ satisfies the conditions of definition 3.1:

Proposition 3.16 *Let Δ, Δ_1, Δ_2, Δ_1', and Δ_2', be worlds of $\mathcal{M}$ such that*

$$\Delta_1 * \Delta_2 \subseteq \Delta \quad \text{and} \quad \Delta_1' * \Delta_2' \subseteq \Delta.$$

The two following conditions are satisfied:

- ◇ *If $I(\ell, \Delta_1) = I(\ell, \Delta_1')$ then $\Delta_2 = \Delta_2'$*
- ◇ *If $I(\ell, \Delta_2) = I(\ell, \Delta_2')$ then $\Delta_1 = \Delta_1'$.*

Proof: The two cases are symmetrical, we show the first part of the proposition:

Assume there is a constant b_i of B such that $I(\ell, \Delta_1) = I(\ell, \Delta_1') = [b_i]$ then, by construction,

$$(\ell = b_i) \in \Delta_1 \quad \text{and} \quad (\ell = b_i) \in \Delta_1'.$$

Consider a sentence f of Δ_2; since $\Delta_1 * \Delta_2 \subseteq \Delta$, $(\ell = b_i; f)$ is a sentence of Δ. By axiom L1, it follows that $\neg(\ell = b_i; \neg f)$ is also in Δ. Therefore, $\neg f$ cannot be in Δ_2', so f belongs to Δ_2'. Hence $\Delta_2 \subseteq \Delta_2'$. By symmetry, we also have $\Delta_2' \subseteq \Delta_2$ and the two sets Δ_2 and Δ_2' are equal. □

S is complete

The completeness of S is now a straightforward consequence of corollary 3.15 and proposition 3.16.

Theorem 3.17 *If a formula f of $\mathcal{L}$ is valid in $\mathcal{C}$, it is a theorem of S.*

Proof: Consider a formula f which is not provable in S and let g be the universal closure of f. g is not provable either, otherwise by Q1 and MP, f could be deduced from g. Let Γ_0 be the set $\{\neg g\}$. Γ_0 is consistent, the construction of section 3.3.3 can be used. This yields a model $\mathcal{M}$ which satisfies $\neg g$ (by corollary 3.15) and which belongs to the class $\mathcal{C}$ (by proposition 3.16). Strictly speaking, $\mathcal{M}$ is a model for the language $\mathcal{L}^+$ but we can easily transform $\mathcal{M}$ to a model for $\mathcal{L}$ by restricting the interpretation function I to symbols of $\mathcal{L}$. The satisfaction of sentences of $\mathcal{L}$ is not changed. Since $\neg g$ is satisfied in a world Δ_0 of $\mathcal{M}$, the sentence g is not valid in $\mathcal{M}$ and f is not valid either. □

Chapter 4

Time intervals

In the previous chapter, we have considered a class $\mathcal{C}$ of models for ITL defined by a constraint on the "length" of worlds. The proof system S has been shown to be complete for this class. In the present chapter, we define a new class of models where worlds are time intervals and the length of intervals is their duration. The two basic ingredients for constructing such models are a notion of temporal domain for defining frames and an abstract measure function assigning a duration to intervals. The class $\mathcal{K}$ of interval models is defined in section 4.1; it is a strict sub-class of $\mathcal{C}$.

In section 4.2 we present a new proof system S' for reasoning about interval models. This system is obtained from S by adding a few axioms expressing properties of the measure function and intervals. We give a few example theorems derived by S'.

In the last section of this chapter, we establish the completeness of the axiomatization. Any formula valid in $\mathcal{K}$ is provable in S'. The proof relies on the completeness of S. Any set of sentences Γ consistent relatively to S' is also consistent relatively to S and is then satisfied by an S-model $\mathcal{M}_0$. We show that, provided the additional axioms of S' are valid in $\mathcal{M}_0$, an interval model of Γ can be constructed from $\mathcal{M}_0$.

4.1 Interval models

4.1.1 Temporal domains and intervals

Intuitively, a temporal interval can be considered as an uninterrupted stretch of time delimited by two instants t and t' such that t' is posterior to t. This assumes that time is a set of instants, equipped with an order relation. Various additional assumptions can be made about the structure of time: the order can be total or partial, dense or discrete, etc. We only assume linear time and we call a particular time representation a *temporal domain*.

Definition 4.1 *A temporal domain is a pair $(T, \leqslant)$ where T is a non-empty set and $\leqslant$ a total order relation on T.*

We will usually denote a temporal domain simply by T, letting the order relation implicit.

Assuming a temporal domain T is given, we define the intervals on T as pairs of elements (t, t') of T such that $t \leqslant t'$. Such pairs are denoted by $[t, t']$. Then we can derive from T a frame (W, R) called an *interval frame* as follows:

- ⋄ W is the set of intervals on T,
- ⋄ R is the ternary relation on W defined by the rule

$$R([t_1, t_1'], [t_2, t_2'], [t, t']) \quad \text{iff} \quad t = t_1, t_1' = t_2, t_2' = t,$$

  for any intervals $[t_1, t_1']$, $[t_2, t_2']$, and $[t, t']$ of W. In other words, an interval $[t, t']$ can be split into any pair of intervals $[t, u]$, $[u, t']$ such that $t \leqslant u \leqslant t'$. This corresponds to the intuitive idea of "chopping" the interval $[t, t']$ in two sub-intervals.

Classic examples of temporal domains are the set $\mathbb{R}^+$ of non-negative real numbers used to model dense time, or the set $\mathbb{N}$ of natural numbers for discrete time.

4.1.2 Measure

Let T be an arbitrary temporal domain and W be the set of intervals on T. We want to assign a length to every interval $[t, t']$ of W. This length will be given by a function m we call a *measure*.

For the two usual temporal domains $T = \mathbb{N}$ or $T = \mathbb{R}^+$, a natural choice for the measure m is to set

$$m[t, t'] \quad = \quad t' - t,$$

hence m is a function from W to T. However, there is no reason to assume that this is always the case, instants and durations are two different concepts and do not have to be represented by elements of the same set. So, in general, we assume that some set D is given whose elements are possible lengths or durations of intervals and m will be a function from W to D.

Constraints on m

In order to capture a "reasonable" notion of measure, the function m has to satisfy a few intuitive properties.

One of them has already been presented in the definition of S-models (cf section 3.1.1). Two distinct prefixes $[t, u]$ and $[t, u']$ or two distinct suffixes $[u, t']$ and $[u', t']$ of an interval $[t, t']$ cannot have the same length.

We also assume that the length of point intervals is null. So, we need a distinguished element 0 of D and we require $m[t, t] = 0$ for any instant $t \in T$.

We require additivity of lengths. We assume that a binary operation $+$ is available on D and that we have $m[t, u] + m[u, t'] = m[t, t']$ for $t \leqslant u \leqslant t'$.

Our final requirement for m is the converse of the previous one. If an interval $[t, t']$ has length $x + y$ then it has a prefix $[t, u]$ of length x and for this u, the suffix $[u, t']$ is of length y.

In summary, m is a function from W to a set D with a binary operation $+$ and a distinguished element 0, and the measure is required to satisfy the four following conditions.

M1: if $m[t, u] = m[t, u']$ then $u = u'$ and
if $m[u, t] = m[u', t]$ then $u = u'$

M2: $m[t, t] = 0$ for any instant $t \in T$.

M3: $m[t, u] + m[u, t'] = m[t, t']$ for $t \leqslant u \leqslant t'$.

M4: if $m[t, t'] = x + y$, there is $u \in T$ such that
$t \leqslant u \leqslant t'$, $m[t, u] = x$, and $m[u, t'] = y$.

Note that combining M1 and M2 implies that only point intervals are of length 0: if $m[t, t'] = 0$ then $t = t'$.

These requirements are generalizations to abstract measures of properties satisfied by the usual notions of lengths of intervals. For example, the natural measure defined by $m[t, t'] = t' - t$ for the temporal domain $T = \mathbb{N}$ is easily seen to satisfy conditions M1 to M4. It is also the case for $T = \mathbb{R}^+$ if D is the set of non-negative reals, but condition M4 does not hold if m is considered as a function from W to $\mathbb{R}$ (for x or y can be negative).

Duration domains

We have assumed that D was equipped with a binary operation $+$ and contained at least one element 0, but so far, no particular assumptions on the behaviour of $+$ or 0 in D have been made. However, if there is a function m from W to D which satisfies M1 to M4 then $+$ and 0 must obey classic algebraic laws.

Indeed, let D be an arbitrary set, 0 an element of D and $+$ a binary operation on D and assume there is a function m from W to D which satisfies the conditions M1 to M4. If m is surjective, it is easy to check that

- $+$ is associative,
- 0 is a neutral element for $+$,
- the left and right cancellation laws hold,
- if $x + y = 0$ then $x = 0$ and $y = 0$.

These properties follow from M1–M4, and the definition of intervals on T.

In general, it is possible that the above properties do not hold everywhere in D. For example, there can be two non-null elements x and y such that

$x + y = 0$ provided m does not assign x or y to any interval $[t, t']$. However, these properties always hold in the sub-algebra $(m(W), +, 0)$ where $m(W) \subseteq D$ is the image of W by m.

Other subsets of D are constrained to satisfy another important property. Consider an arbitrary instant t of T and let E be the subset of D defined by

$$x \in E \quad \text{iff there is } t' \geqslant t \text{ s.t.} \quad m[t, t'] = x.$$

In other words, E is the set of measures of the intervals $[t, t']$ of W. Let $x = m[t, t']$ and $y = m[t, t'']$ be two elements of E. Since T is totally ordered, we have either $x + m[t', t''] = y$ or $y + m[t'', t'] = x$. Hence for any two elements x and y of E there is some z of D such that $x + z = y$ or $y + z = x$.

Symmetrically, if x and y are measures or two intervals $[t', t]$ and $[t'', t]$ then there is a z of D such that $z + x = y$ or $z + y = x$.

Hence, the existence of a function m from W to D which satisfies conditions M1–M4 imposes some constraints on the algebra $(D, +, 0)$. The properties above must hold in subsets of D. We will only consider structures $(D, +, 0)$ where these properties are satisfied on D as a whole. Such structures will be called *duration domains*.

Duration domains can then be characterized in first order logic as the models of the following formulas:

D1: $\quad (x + y) + z = x + (y + z)$

D2: $\quad x + 0 = x$
$\qquad\;\; 0 + x = x$

D3: $\quad x + y = x + z \Rightarrow y = z$
$\qquad\;\; y + x = z + x \Rightarrow y = z$

D4: $\quad x + y = 0 \Rightarrow x = 0 \wedge y = 0$

D5: $\quad (\exists z)(x + z = y \vee y + z = x)$
$\qquad\;\; (\exists z)(z + x = y \vee z + y = x).$

Measure functions for T can now be defined precisely as the functions from the set of intervals W to some duration domain D and such that the four constraints M1 to M4 are satisfied.

A similar axiomatic approach for defining time delays and associated operations can be found in [22]. All the traditional ways of assigning a length to intervals conform to the definition of duration domains. In the duration calculus or in dense ITL lengths of intervals are positive real numbers [14, 8] and it is clear that all the axioms D1 to D5 are satisfied. For discrete ITL, lengths are natural numbers and D1–D5 also hold [21].

4.1.3 The class $\mathcal{K}$

The basic notions of time domain, measure and duration domain are fundamental in the study of existing systems of ITL used for real-time reasoning. Combined together, the three elements allow us to define the class of interval models.

Languages for such models are required to contain, in addition to the flexible constant ℓ, two rigid symbols $+$ and 0. These symbols will be interpreted as the addition and neutral element in a duration domain and provide a minimal set of operators for expressing real-time constraints. An ITL-language $\mathcal{L}$ which includes these three symbols, is called an *interval language*.

Let T be a temporal domain, m a measure for T with duration domain $(D, +, 0)$ and $\mathcal{L}$ an arbitrary interval language. The three components T, m, and D can serve as a basis for constructing models $\mathcal{M}$ for $\mathcal{L}$ in the following way:

- ◇ the frame of $\mathcal{M}$ is the frame (W, R) defined by T,
- ◇ the domain of $\mathcal{M}$ is the set D,
- ◇ the interpretation in $\mathcal{M}$ of the symbols ℓ, $+$, and 0 is such that[1]

$$\begin{aligned} I(\ell, [t, t']) &= m[t, t'], \\ I(0, [t, t']) &= 0, \\ I(+, [t, t']) &= +, \end{aligned}$$

  for any interval $[t, t']$ of W.

A model constructed in this way is called an *interval model*. The class of interval models is denoted by $\mathcal{K}$.

Note that the interpretation of symbols of $\mathcal{L}$ other than ℓ, $+$ or 0 is free. There can be different interval models $\mathcal{M}$ constructed from the same basis and for a same language $\mathcal{L}$.

For an interval model $\mathcal{M}$ the semantics can be rephrased in terms of the underlying time domain and measure. In particular, for chop formulas, the rule can be rewritten:

$$[t, t'], v \models (f_1; f_2) \quad \text{iff} \qquad \text{there is } u \in T, \quad \begin{cases} t \leqslant u \leqslant t' \\ [t, u], v \models f_1 \\ [u, t'], v \models f_2. \end{cases}$$

This is how the semantics of ITL or the duration calculus is traditionally presented [14, 21]; possible worlds are not mentioned and the semantics is given directly in terms of intervals. In the two cases, time domains are fixed *a priori*. In the duration calculus, time is represented by $\mathbb{R}^+$. In traditional ITL the temporal domain is $T = \mathbb{N}$ [21] and a a densed-time semantics is also proposed in [14] (with $T = \mathbb{R}^+$). The standard models of ITL and the duration calculus are then included in our notion of interval models.

[1] On the left side of the equations, $+$ and 0 are symbols of $\mathcal{L}$, and the right side the same notations are used for the addition operation and the zero element of D.

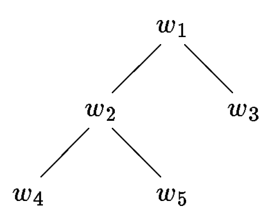
4.1.4 Formulas valid in interval models

By definition of measures, it is clear that any interval model is also an S-model; this follows immediately from M1. The class $\mathcal{K}$ is then a sub-class of $\mathcal{C}$ and all the formulas valid in $\mathcal{C}$ are also valid in $\mathcal{K}$. On the other hand, the construction of a frame from a temporal domain induces new properties of the accessibility relation R. As a consequence many sentences valid in interval models are not valid in $\mathcal{C}$. In other words, $\mathcal{K}$ is a strict sub-class of $\mathcal{C}$.

For example, it is easy to check that, for interval models, chop is associative: for any formulas f, g, h, the following equivalence is valid in $\mathcal{K}$

$$((f;g);h) \Leftrightarrow (f;(g;h)).$$

But, it is not difficult to construct an S-model in which this formula is not valid. For example, let W be a set of five worlds $w_1, \ldots, w_5$ and let R be the ternary relation on W such that $R(w_2, w_3, w_1)$ and $R(w_4, w_5, w_2)$ only. Then (W, R) forms a frame which can be depicted as follows.



Any ITL model built from this frame is necessarily an S-model since the two worlds w_1 and w_2 can only de decomposed in one way and the other worlds cannot be decomposed at all. But, in any such model chop is not associative: if f is a tautology and v any valuation,

$$w_1, v \models ((f;f);f) \quad \text{but} \quad w_1, v \not\models (f;(f;f)).$$

and then

$$w_1, v \not\models ((f;f);f) \Leftrightarrow (f;(f;f)).$$

Since the operations $+$ on D and the element 0 of D are represented by rigid symbols in an interval model, the first order formulas D1–D5 which are satisfied by any duration domain are also valid in any interval model. More generally, any first order formula satisfied by all the duration domains, i.e. any formula f which can be proved in first order logic with equality from D1–D5, is valid in $\mathcal{K}$.

Various formulas which combine additions of lengths and chop are also valid in interval models. For example, the following formulas hold due to the constraints M1–M4 on measures and to properties of interval frames:

$$(\ell = x; \ell = y) \Leftrightarrow \ell = x + y, \quad f \Leftrightarrow (f; \ell = 0), \quad f \Leftrightarrow (\ell = 0; f)$$

4.2 A proof system for interval models

4.2.1 New axioms

In order to reason formally about intervals, we extend the system S by adding new axioms expressing properties of interval frames and relations between lengths and chop. These axioms are present in various existing proof systems proposed both for ITL or the duration calculus. The resulting system is called S'.

The new modal axioms are the following:

$$\text{A2:}\quad ((f;g);h) \Leftrightarrow (f;(g;h))$$

$$\text{L2:}\quad \ell = x + y \Leftrightarrow (\ell = x; \ell = y)$$

$$\text{L3:}\quad \begin{array}{l} f \Rightarrow (f; \ell = 0) \\ f \Rightarrow (\ell = 0; f). \end{array}$$

A2 is the associativity of chop, L2 corresponds to the additivity of measure and L3 expresses that an interval can always be split into itself and a point interval.

The other new axioms of S' are the formulas D1–D5 describing properties of the addition in duration domains.

$$\text{D1:}\quad (x+y)+z = x+(y+z)$$

$$\text{D2:}\quad \begin{array}{l} x + 0 = x \\ 0 + x = x \end{array}$$

$$\text{D3:}\quad \begin{array}{l} x + y = x + z \Rightarrow y = z \\ y + x = z + x \Rightarrow y = z \end{array}$$

$$\text{D4:}\quad x + y = 0 \Rightarrow x = 0 \wedge y = 0$$

$$\text{D5:}\quad \begin{array}{l} (\exists z)(x + z = y \vee y + z = x) \\ (\exists z)(z + x = y \vee z + y = x). \end{array}$$

4.2.2 Soundness and examples of theorems

Since all the new axioms of S' are valid in interval models, the proof system is sound. Any formula f provable in S' is valid in $\mathcal{K}$. As before, $\vdash_{S'} f$ will be used to denote that f is a theorem of S'.

As an example, we can show that for any formula f of an interval language, the equivalence $f \Leftrightarrow (f; \ell = 0)$ is a theorem of S'. The implication

$$f \Rightarrow (f; \ell = 0)$$

is axiom L2 and the reverse implication can be derived as follows:

1	$(f; \ell = 0) \Rightarrow \neg(\neg f; \ell = 0)$	L1
2	$\neg f \Rightarrow (\neg f; \ell = 0)$	L3
3	$\neg(\neg f; \ell = 0) \Rightarrow f$	PC, 2
4	$(f; \ell = 0) \Rightarrow f$	PC, 1, 3.

Other important theorems can be derived by first order calculus from the axioms D1–D5, such as the three following:

O1: $(\exists z)(x + z = x)$

O2: $(\exists z)(x + z = y) \wedge (\exists z)(y + z = x) \Rightarrow x = y$

O3: $(\exists z)(x + z = y) \wedge (\exists z)(y + z = u) \Rightarrow (\exists z)(x + z = u)$.

These three theorems show that D can always be equipped with an order relation $\leqslant$ defined by $x \leqslant y$ if there is $z \in D$ such that $x + z = y$. This relation is also a total order, by axiom D5. We will call it the *natural order* on D.

4.3 Completeness

In this section, we show the completeness of S'. If $\mathcal{L}$ is an arbitrary interval language then any formula f of $\mathcal{L}$ which is valid in interval models is a theorem of S'. As in the case of S, the principle is to show that any set Γ_0 of sentences of $\mathcal{L}$ which is consistent with respect to S' is satisfied by an interval model. Since S' is an extension of S, the model construction of section 3.3.3 can be applied to Γ_0 and yields an S-model $\mathcal{M}_0$ of Γ_0. We can construct from $\mathcal{M}_0$ an interval model $\mathcal{M}$ which also satisfies Γ_0.

For this, we first study properties of $\mathcal{M}_0$ due to the validity of the new axioms A2 and L3. In a second step, we will construct a temporal domain T based on $\mathcal{M}_0$. An essential property of T is the existence of a mapping μ from intervals of T to worlds of $\mathcal{M}_0$ which preserve the frame structure (μ is a homomorphism). Finally, we define an interval model $\mathcal{M}$ based on T and the fact that $\mathcal{M}$ is a model of Γ_0 is an easy consequence of the properties of μ.

4.3.1 The model $\mathcal{M}_0$

Let $\mathcal{L}$ be an interval language and Γ_0 a set of sentences of $\mathcal{L}$. Definition 3.2 extends in a natural way to the system S' so we say that Γ_0 is consistent with respect to S' if there is no finite subset $\{f_1, \ldots, f_n\}$ $(n \geqslant 1)$ of Γ_0 such that $\vdash_{S'} \neg(f_1 \wedge \ldots \wedge f_n)$. The notion of maximal consistent sets extends similarly.

The model construction given in 3.3.3 is based on consistent and maximal consistent sets with respect to S. It requires that all the instances of axioms A1, L1, R, and B be present in any consistent set. Since S' is an extension of S, the model construction also works for sets of sentences consistent with respect to S'.

As before, $\mathcal{L}^+$ denotes a new interval language obtained by adding to $\mathcal{L}$ a new set of rigid constants B. If Γ_0 is a consistent set of sentences with respect to S' then it can be extended to a set $\Gamma_0^\star$ of sentences of $\mathcal{L}^+$ which is maximal consistent with respect to S' and has witnesses in B. We denote by Σ_0 the set of rigid sentences of $\Gamma_0^\star$. The construction of section 3.3.3 yields an S-model $\mathcal{M}_0 = (W_0, R_0, D_0, I_0)$ where

- ◇ W_0 is the set of sentences Δ such that Δ is maximal consistent with respect to S' and has witnesses in B and such that Σ_0 is included in Δ.
- ◇ R_0 is the relation defined by

$$R_0(\Delta_1, \Delta_2, \Delta) \quad \text{iff} \quad \Delta_1 * \Delta_2 \subseteq \Delta.$$

- ◇ D_0 is the set of equivalence classes of the relation $\equiv$ on B defined by

$$b_i \equiv b_j \quad \text{iff} \quad (b_i = b_j) \in \Sigma_0.$$

$\Gamma_0^\star$ is one of the worlds of W_0 and in this world all the sentences of Γ_0 are satisfied.

4.3.2 Properties of $\mathcal{M}_0$

In the remainder of this section, consistent always mean consistent with respect to S'.

All the instances of axioms A2, L2, and L3 are present in any world Δ of W_0 since they must be in any maximal consistent set. This imposes various properties on the accessibility relation R_0.

Lemma

In order to establish these properties, we will need the following lemma. Recall that the two functions δ_1 and δ_2 are defined by:

$$\begin{aligned} \delta_1(\Gamma, \Gamma_1) &= \{\neg g | \neg(f;g) \in \Gamma, f \in \Gamma_1\}, \\ \delta_2(\Gamma, \Gamma_2) &= \{\neg f | \neg(f;g) \in \Gamma, g \in \Gamma_2\}. \end{aligned}$$

for arbitrary sets of sentences Γ, Γ_1, and Γ_2.

Lemma 4.2 *Let Δ, Δ_1, and Δ_2 be three worlds of W_0 and Γ_1 and Γ_2 be two maximal consistent sets of sentences of $\mathcal{L}^+$.*

- ◇ *If $\delta_1(\Delta, \Delta_1) \subseteq \Gamma_2$ then Γ_2 belongs to W_0 and $R_0(\Delta_1, \Gamma_2, \Delta)$.*
- ◇ *If $\delta_2(\Delta, \Delta_2) \subseteq \Gamma_1$ then Γ_1 belongs to W_0 and $R_0(\Gamma_1, \Delta_2, \Delta)$.*

Proof: For the first half of the lemma, assume $\delta_1(\Delta, \Delta_1) \subseteq \Gamma_2$ and let f and g be two sentences of Δ_1 and Γ_2, respectively. If $\neg(f;g)$ is in Δ then by definition of δ_1, $\neg f$ must be in Γ_2 this yields a contradiction. Hence we have $\Delta_1 * \Gamma_2 \subseteq \Delta$, that is $R_0(\Delta_1, \Gamma_2, \Delta)$. By lemma 3.12 this implies that Γ_2 is a world of W_0. The proof is similar for the other half of the lemma. □

Associativity

The validity of A2 in $\mathcal{M}_0$ implies the following property of R_0.

Proposition 4.3 *Given four worlds Δ, Δ_1, Δ_2, and Δ_3 of W_0, the two following propositions are equivalent.*

- ◇ *There is a world Δ' such that $R_0(\Delta_1, \Delta_2, \Delta')$ and $R_0(\Delta', \Delta_3, \Delta)$.*
- ◇ *There is a world Δ'' such that $R_0(\Delta_1, \Delta'', \Delta)$ and $R_0(\Delta_2, \Delta_3, \Delta'')$.*

Proof: We show that the first part of the proposition implies the second. The converse implication follows by symmetry.

Assume there is a world Δ' of W_0 such that

$$R_0(\Delta_1, \Delta_2, \Delta') \quad \text{and} \quad R_0(\Delta', \Delta_3, \Delta),$$

that is,

$$\Delta_1 * \Delta_2 \subseteq \Delta' \quad \text{and} \quad \Delta' * \Delta_3 \subseteq \Delta.$$

By construction of W_0 there are constants b_1, b_2, and b_3 of B such that

$$(\ell = b_1) \in \Delta_1, \quad (\ell = b_2) \in \Delta_2, \quad \text{and} \quad (\ell = b_3) \in \Delta_3.$$

In order to establish the existence of Δ'', it is sufficient to show that the following set of sentences is consistent

$$A \quad = \quad \{(\ell = b_2; \ell = b_3)\} \cup \delta_1(\Delta, \Delta_1).$$

Indeed, if A is consistent, then it can be extended to a maximal consistent set Δ'' by Lidenbaum's lemma. By lemma 4.2, Δ'' is a world of W_0 and $R_0(\Delta_1, \Delta'', \Delta)$. If h_2 and h_3 are two sentences of Δ_2 and Δ_3, respectively, then we have

$$(\ell = b_1; h_2) \in \Delta' \quad \text{and} \quad ((\ell = b_1; h_2); h_3) \in \Delta.$$

By axioms A2 and L1, it follows that

$$(\ell = b_1; (h_2; h_3)) \in \Delta \quad \text{and} \quad \neg(\ell = b_1; \neg(h_2; h_3)) \in \Delta.$$

Then $\neg\neg(h_2; h_3)$ is in $\delta_1(\Delta, \Delta_1)$ and this implies that $(h_2; h_3) \in \Delta''$. Hence, we have $\Delta_2 * \Delta_3 \subseteq \Delta''$, that is, $R_0(\Delta_2, \Delta_3, \Delta'')$.

In order to show that A is consistent, consider n sentences $f'_1, \ldots, f'_n$ of $\delta_1(\Delta, \Delta_1)$. By definition of δ_1 there are formulas $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ such that, for $i = 1, \ldots, n$,

- ◇ f'_i is $\neg f_i$,
- ◇ g_i belongs to Δ_1,
- ◇ $\neg(g_i; f_i)$ belongs to Δ.

Let g be the conjunction $g_1 \wedge \ldots \wedge g_n$. g is in Δ_1 and for all i, $\neg(g; f_i)$ is in Δ (cf. lemma 3.10). On the other hand, since $\Delta_1 * \Delta_2 \subseteq \Delta'$ and $\Delta' * \Delta_3 \subseteq \Delta$, we have

$$(g; \ell = b_2) \in \Delta' \quad \text{and} \quad ((g; \ell = b_2); \ell = b_3) \in \Delta.$$

then, by A2,

$$(g; (\ell = b_2; \ell = b_3)) \in \Delta.$$

Using A1 repeatedly yields

$$\vdash_{S'} (g; (\ell = b_2; \ell = b_3)) \wedge \neg(g; f_1) \wedge \ldots \wedge \neg(g; f_n) \Rightarrow \\ (g; (\ell = b_2; \ell = b_3) \wedge \neg f_1 \wedge \ldots \wedge \neg f_n),$$

then the sentence $(g; (\ell = b_2; \ell = b_3) \wedge \neg f_1 \wedge \ldots \wedge \neg f_n)$ is also in Δ. Hence, for arbitrary $f'_1, \ldots, f'_n$ of $\delta_1(\Delta, \Delta_1)$ there is a sentence g such that

$$(g; (\ell = b_2; \ell = b_3) \wedge f'_1 \wedge \ldots f'_n) \in \Delta.$$

We cannot have

$$\vdash_{S'} \neg((\ell = b_2; \ell = b_3) \wedge f'_1 \wedge \ldots f'_n)$$

otherwise, the necessity rule would yield

$$\vdash_{S'} \neg(g; (\ell = b_2; \ell = b_3) \wedge f'_1 \wedge \ldots f'_n)$$

and this would contradict the consistency of Δ. Hence, for any $f'_1, \ldots, f'_n$ of $\delta_1(\Delta, \Delta_1)$ we have,

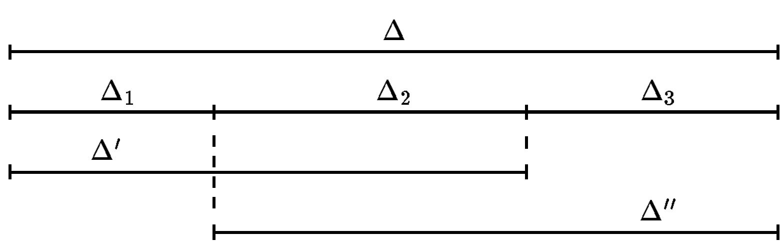
$$\not\vdash_{S'} \neg((\ell = b_2; \ell = b_3) \wedge f'_1 \wedge \ldots f'_n).$$

This means that A is consistent. $\square$

The latter property states a form of associativity of $*$ in W_0:

$$(\Delta_1 * \Delta_2) * \Delta_3 \subseteq \Delta \quad \text{iff} \quad \Delta_1 * (\Delta_2 * \Delta_3) \subseteq \Delta.$$

Anticipating on further results, we will represent Δ, Δ_1, Δ_2, and Δ_3 as if they were intervals. Property 4.3 can then be depicted as follows.



Note also that since $\mathcal{M}_0$ is an S-model the two worlds Δ' and Δ'', if they exist, are unique.

Reflexivity

The next property of R_0 is a consequence of axiom L3: any world Δ can be split into itself and a world of length 0.

Proposition 4.4 *For any Δ in W_0 there are two worlds Δ_1 and Δ_2 such that*

- $R_0(\Delta_1, \Delta, \Delta)$ *and* $(\ell = 0) \in \Delta_1$,
- $R_0(\Delta, \Delta_2, \Delta)$ *and* $(\ell = 0) \in \Delta_2$.

Furthermore Δ_1 and Δ_2 are unique.

Proof: The two cases are symmetrical, we only show the existence of Δ_1.

The proof is very similar to that of proposition 4.3. We first show that the set A defined by

$$A = \{(\ell = 0)\} \cup \delta_2(\Delta, \Delta)$$

is consistent. For this, let $f'_1, \ldots, f'_n$ be n sentences of $\delta_2(\Delta, \Delta)$. There are then $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ such that,

- f'_i is $\neg f_i$,
- g_i and $\neg(f_i; g_i)$ belong to Δ.

Let g be the conjunction $g_1 \wedge \ldots \wedge g_n$ then we have, as above,

$$g \in \Delta \quad \text{and} \quad \neg(f_i; g) \in \Delta.$$

By L3, the sentence $(\ell = 0; g)$ must also be in Δ and, by the same mechanism as in the previous proposition,

$$(\ell = 0 \wedge \neg f_1 \wedge \ldots \wedge \neg f_n; g) \in \Delta.$$

Then we cannot have

$$\vdash_{S'} \neg(\ell = 0 \wedge f'_1 \wedge \ldots \wedge f'_n),$$

and A is consistent.

Now let Δ_1 be a maximal consistent extension of A; we have $(l = 0) \in \Delta_1$ and, by lemma 4.2, Δ_1 is a world of W_0 and $\Delta_1 * \Delta \subseteq \Delta$. Since $\mathcal{M}_0$ is an S-model, Δ_1 is unique. □

This proposition can be interpreted as a form of "reflexivity" of R_0: any world Δ is both its own "prefix" and its own "suffix".

4.3.3 Temporal domain obtained from $\mathcal{M}_0$

The two previous propositions show that $\mathcal{M}_0$ shares two properties with interval models. These two properties only required the validity of A2 and L3 in $\mathcal{M}_0$. In this part we show further similarities between $\mathcal{M}_0$ and interval models. We construct from $\mathcal{M}_0$ a temporal domain T in such a way that the interval frame defined by T is homomorphic to a sub-frame of $\mathcal{M}_0$. This relies on the properties of duration domains and on axiom L2.

Definition of T

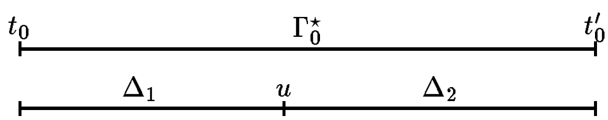
By construction of $\mathcal{M}_0$, there is a world $\Gamma_0^\star$ of W_0 which satisfies Γ_0. The satisfaction of formulas of $\mathcal{L}^+$ in $\Gamma_0^\star$ does not depend on the worlds of W_0 which are not related by R to $\Gamma_0^\star$. So we can restrict our attention to the sub-frame[2] defined by the worlds related to $\Gamma_0^\star$.

Our objective is to construct a temporal domain T such that every interval of W can be associated to a world in this sub-frame by a mapping μ preserving the properties of R_0. We want

- $R_0(\mu[t,t'],\mu[t',t''],\mu[t,t''])$ for all t, t' and t'' of T such that $t \leqslant t' \leqslant t''$,
- conversely, if $R_0(\Delta_1,\Delta_2,\mu[t,t'])$ there must be a point u of T such that $t \leqslant u \leqslant t'$, $\mu[t,u]=\Delta_1$, and $\mu[u,t']=\Delta_2$.

If such a mapping exists, it is not hard to construct an interval model $\mathcal{M}$ based on T and such that an interval $[u,u']$ satisfies the same formulas in $\mathcal{M}$ as the world $\mu[u,u']$ in $\mathcal{M}_0$.

Starting from this idea, we want μ to map an interval of W, say $[t_0,t_0']$, to the world $\Gamma_0^\star$. Then for any pair of worlds (Δ_1,Δ_2) such that $R_0(\Delta_1,\Delta_2,\Gamma_0^\star)$ there must be a unique instant u in T such that $t_0 \leqslant u \leqslant t_0'$ and the two sub-intervals $[t_0,u]$ and $[u,t_1']$ are associated with Δ_1 and Δ_2 respectively. Conversely, every instant u such that $t_0 \leqslant u \leqslant t_0'$ uniquely defines two worlds Δ_1 and Δ_2 such that $R_0(\Delta_1,\Delta_2,\Gamma_0^\star)$ as illustrated in the following figure.



Hence, there must be a bijection between the pairs of worlds (Δ_1,Δ_2) of W_0 such that $R_0(\Delta_1,\Delta_2,\Gamma_0^\star)$ and the set of instants u of T such that $t_0 \leqslant u \leqslant t_0'$.

There are different possibilities to find a temporal domain T satisfying the latter requirement. A possible choice is to define the set T as exactly the set of pairs (Δ_1,Δ_2) such that $R_0(\Delta_1,\Delta_2,\Gamma_0^\star)$, that is,

$$T \quad = \quad \{\, (\Delta_1,\Delta_2) \mid \Delta_1 * \Delta_2 \subseteq \Gamma_0^\star \,\}.$$

A relation $\leqslant$ can be defined on T by

$$(\Delta_1,\Delta_2) \leqslant (\Delta_1',\Delta_2') \quad \text{if there are } b_i \text{ and } b_j \text{ in } B \text{ s.t.} \quad \begin{cases} (\ell = b_i) \ \in \Delta_1 \\ (\ell = b_i + b_j) \ \in \Delta_1', \end{cases}$$

and the following property shows that $(T,\leqslant)$ is actually a temporal domain.

[2] Formally, this is the frame (W_1,R_1) where W_1 is the smallest subset of W_0 containing $\Gamma_0^\star$ and such that, whenever $\Delta \in W_1$, all the worlds Δ_1 and Δ_2 satisfying $R_0(\Delta_1,\Delta_2,\Delta)$ are also in W_1 and R_1 is the restriction of R_0 to W_1.

Proposition 4.5 *The relation $\leqslant$ is a total order on T.*

Proof: Let (Δ_1, Δ_2) and (Δ'_1, Δ'_2) be two elements of T. The previous definition is easily seen to be equivalent to the following relation

$$(\Delta_1, \Delta_2) \leqslant (\Delta'_1, \Delta'_2) \quad \text{iff there are } b_i \text{ and } b_j \text{ s.t.} \quad \begin{cases} (\ell = b_i) & \in \Delta_1 \\ (\ell = b_j) & \in \Delta'_1, \\ (\exists z)(b_i + z = b_j) & \in \Sigma_0. \end{cases}$$

By theorems O1, O2, and O3, this implies that $\leqslant$ is an order relation on T and, by axiom D5, it is total. Note that this implicitly relies also on axioms D1-D4 which are used to derive O1, O2, and O3. □

Informally, the definition of $\leqslant$ simply means that the instant (Δ_1, Δ_2) is anterior to the instant (Δ'_1, Δ'_2) if and only if the length of Δ_1 is smaller than the length of Δ'_1 for the natural order on D.

Properties of T

The temporal domain T has been chosen to satisfy a necessary condition. We will now define a mapping μ from the set W of intervals of T to the set W_0 of worlds of $\mathcal{M}_0$ and we will show that μ behaves as expected. The fundamental property is the following.

Proposition 4.6 *Let (Δ_1, Δ_2) and (Δ'_1, Δ'_2) be two elements of T such that $(\Delta_1, \Delta_2) \leqslant (\Delta'_1, \Delta'_2)$ then there is a unique world Δ of W_0 such that*

$$R_0(\Delta_1, \Delta, \Delta'_1) \quad \text{and} \quad R_0(\Delta, \Delta'_2, \Delta_2).$$

Proof: There are two constants b_1 and b_2 of B such that

$$(\ell = b_1) \ \in \Delta_1 \quad \text{and} \quad (\ell = b_2) \ \in \Delta_2.$$

Similarly, there are b'_1 and b'_2 such that

$$(\ell = b'_1) \ \in \Delta'_1 \quad \text{and} \quad (\ell = b'_2) \ \in \Delta'_2.$$

Since $(\Delta_1, \Delta_2) \leqslant (\Delta'_1, \Delta'_2)$, there is also a constant b of B such that

$$(b_1 + b = b'_1) \ \in \Sigma_0$$

and by construction of $\mathcal{M}_0$ the sentence $(b_1 + b = b'_1)$ belongs to all the worlds of W_0, in particular to Δ'_1.

The proof follows the same principle as in propositions 4.3 and 4.4. We define a set of sentences A as follows:

$$A \ = \ \{\ell = b\} \cup \delta_1(\Delta'_1, \Delta_1) \cup \delta_2(\Delta_2, \Delta'_2),$$

then we show that A is consistent. The set Δ can be taken to be a maximal consistent extension of A and it will satisfy the two required conditions.

We first prove that A is consistent. Consider n formulas $\neg f_1, \ldots, \neg f_n$ of $\delta_1(\Delta_1', \Delta_1)$, and m formulas $\neg g_1, \ldots, \neg g_m$ of $\delta_2(\Delta_2, \Delta_2')$. There are sentences $f_1', \ldots, f_n'$ and $g_1', \ldots, g_m'$ such that

$$f_i' \in \Delta_1, \quad \neg(f_i'; f_i) \in \Delta_1', \quad g_j' \in \Delta_2', \quad \text{and} \quad \neg(g_j; g_j') \in \Delta_2,$$

for all i in $1, \ldots, n$ and all j in $1, \ldots, m$. Let f' and g' be the two sentences $(f_1' \wedge \ldots \wedge f_n')$ and $(g_1' \wedge \ldots \wedge g_m')$. As Δ_1 and Δ_2' are maximal consistent sets, we have

$$(f' \wedge \ell = b_1) \in \Delta_1 \quad \text{and} \quad (g' \wedge \ell = b_2') \in \Delta_2'.$$

Also, as in proposition 4.3 and 4.4, we have, for all i and all j,

$$\neg(f' \wedge \ell = b_1 \ ; \ f_i) \in \Delta_1' \quad \text{and} \quad \neg(g_j \ ; \ g' \wedge \ell = b_2') \in \Delta_2.$$

By definition of T,

$$\Delta_1 * \Delta_2 \subseteq \Gamma_0^\star \quad \text{and} \quad \Delta_1' * \Delta_2' \subseteq \Gamma_0^\star,$$

this implies that

$$(f' \wedge \ell = b_1 \ ; \ \neg(g_1; g' \wedge \ell = b_2') \wedge \ldots \wedge \neg(g_m; g' \wedge \ell = b_2')) \quad \in \quad \Gamma_0^\star \qquad (4.1)$$

and

$$(\neg(f' \wedge \ell = b_1; f_1) \wedge \ldots \wedge \neg(f' \wedge \ell = b_1; f_n) \ ; \ g' \wedge \ell = b_2') \quad \in \quad \Gamma_0^\star. \qquad (4.2)$$

On the other hand,

$$\vdash_{S'} \ell = b_1' \ \wedge \ b_1 + b = b_1' \ \Rightarrow \ \ell = b_1 + b$$

and by axiom L2,

$$\vdash_{S'} \ell = b_1 + b \ \Rightarrow \ (\ell = b_1; \ell = b).$$

It follows that the sentence $(\ell = b_1; \ell = b)$ belongs to Δ_1'. Then we have

$$((\ell = b_1; \ell = b) \ ; \ g' \wedge \ell = b_2') \quad \in \quad \Gamma_0^\star$$

and, by A2,

$$(\ell = b_1 \ ; \ (\ell = b; g' \wedge \ell = b_2')) \quad \in \quad \Gamma_0^\star. \qquad (4.3)$$

At this point, we need theorem T5 established in section 3.2.4:

$$\text{T5:} \quad (h_1 \wedge \ell = x; h_2) \wedge (\ell = x; h_3) \Rightarrow (h_1 \wedge \ell = x; h_2 \wedge h_3).$$

From this theorem and relations 4.1 and 4.3, it follows that

$$\begin{aligned}&(f' \wedge \ell = b_1 \ ; \\ &\neg(g_1; g' \wedge \ell = b_2') \wedge \ldots \wedge \neg(g_m; g' \wedge \ell = b_2') \wedge (\ell = b; g' \wedge \ell = b_2')) \ \in \Gamma_0^\star.\end{aligned}$$

But, by iterated applications of A1,

$$\begin{aligned}\vdash_{S'} \ &\neg(g_1; g' \wedge \ell = b_2') \wedge \ldots \wedge \neg(g_m; g' \wedge \ell = b_2') \\ &\wedge (\ell = b; g' \wedge \ell = b_2') \ \Rightarrow \ (\ell = b \wedge \neg g_1 \wedge \ldots \wedge \neg g_m; g' \wedge \ell = b_2'),\end{aligned}$$

so, by Mono,

$$(f' \wedge \ell = b_1 \ ; \ (\ell = b \wedge \neg g_1 \wedge \ldots \wedge \neg g_m; g' \wedge \ell = b'_2)) \ \in \Gamma_0^\star.$$

and, by A2,

$$((f' \wedge \ell = b_1; \ell = b \wedge \neg g_1 \wedge \ldots \wedge \neg g_m) \ ; \ g' \wedge \ell = b'_2) \ \in \Gamma_0^\star. \tag{4.4}$$

We now use theorem T6 (section 3.2.4):

$$\text{T6:} \quad (h_1; h_2 \wedge \ell = x) \wedge (h_3; h_2 \wedge \ell = x) \Rightarrow (h_1 \wedge h_3; h_2 \wedge \ell = x).$$

From T6 and relations 4.2 and 4.4, the sentence

$$\begin{aligned}&((f' \wedge \ell = b_1; \ell = b \wedge \neg g_1 \wedge \ldots \wedge \neg g_m) \wedge \\ &\quad \neg(f' \wedge \ell = b_1; f_1) \wedge \ldots \wedge \neg(f' \wedge \ell = b_1; f_n) \ ; \ g' \wedge \ell = b'_2)\end{aligned}$$

belongs to $\Gamma_0^\star$. Using once again A1 and Mono yields

$$((f' \wedge \ell = b_1 \ ; \ \ell = b \wedge \neg g_1 \wedge \ldots \wedge \neg g_m \wedge \neg f_1 \wedge \ldots \wedge \neg f_n) \ ; \ g' \wedge \ell = b'_2) \ \in \ \Gamma_0^\star.$$

As a consequence, the sentence

$$\ell = b \wedge \neg g_1 \wedge \ldots \wedge \neg g_m \wedge \neg f_1 \wedge \ldots \wedge \neg f_n$$

is consistent, otherwise two applications of the necessity rule N would yield a contradiction. This shows that A is consistent.

Now let Δ be a maximal consistent set which includes A. By lemma 4.2, since both

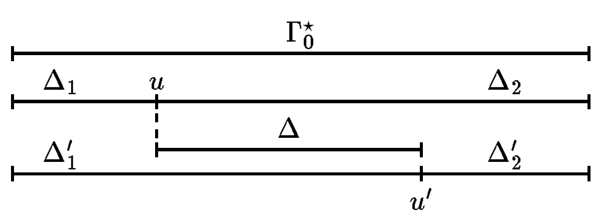
$$\delta_1(\Delta'_1, \Delta_1) \subseteq \Delta \quad \text{and} \quad \delta_2(\Delta_2, \Delta'_2) \subseteq \Delta,$$

the set Δ is a world of W_0 and

$$R_0(\Delta_1, \Delta, \Delta'_1) \quad \text{and} \quad R_0(\Delta, \Delta'_2, \Delta_2).$$

Uniqueness is due to the fact that $\mathcal{M}_0$ is an S-model (cf. proposition 3.16). □

The configuration of the worlds involved of this proposition is illustrated by the following figure.



For two elements $u = (\Delta_1, \Delta_2)$ and $u' = (\Delta_1', \Delta_2')$ of T such that $u \leqslant u'$, Δ is the unique world of W_0 such that

$$\Delta_1 * \Delta \subseteq \Delta_1' \quad \text{and} \quad \Delta * \Delta_2' \subseteq \Delta_2.$$

We can then define a function μ from the set W of intervals $[u, u']$ of T to the set of worlds W_0 such that $\mu[u, u']$ is the world Δ given by proposition 4.6.

The two following properties establish a close link between the two accessibility relations R_0 and R. The first one means that μ is a homomorphism from (W, R) to (W_0, R_0).

Proposition 4.7 *Given three points u, u', and u'' of T such that $u \leqslant u' \leqslant u''$ (i.e. $R([u,u'],[u',u''],[u,u''])$ then $R_0(\mu[u,u'], \mu[u',u''], \mu[u,u''])$.*

Proof: The points u, u', and u'' are three pairs of worlds (Δ_1, Δ_2), (Δ_1', Δ_2'), and (Δ_1'', Δ_2'') respectively and we have, by definition of μ,

- $R_0(\Delta_1, \mu[u, u'], \Delta_1')$,
- $R_0(\Delta_1', \mu[u', u''], \Delta_1'')$,
- $R_0(\Delta_1, \mu[u, u''], \Delta_1'')$,

that is,

$$\Delta_1 * \mu[u, u'] \subseteq \Delta_1', \quad \Delta_1' * \mu[u', u''] \subseteq \Delta_1'', \quad \text{and} \quad \Delta_1 * \mu[u, u''] \subseteq \Delta_1''.$$

We have to show that $\mu[u, u'] * \mu[u', u''] \subseteq \mu[u, u'']$. Let then f and g be two sentences of $\mu[u, u']$ and $\mu[u', u'']$ respectively. There is a constant b of B such that $(\ell = b)$ belongs to Δ_1 and then

$$(\ell = b; f) \in \Delta_1' \quad \text{and} \quad ((\ell = b; f); g) \in \Delta_1''.$$

By A2, this implies

$$(\ell = b; (f; g)) \in \Delta_1''$$

and, by L1,

$$\neg(\ell = b; \neg(f; g)) \ \in \Delta_1''$$

Since $(\ell = b) \in \Delta_1$ and $\Delta_1 * \mu[u, u''] \subseteq \Delta_1''$, this means that $(f; g)$ must be in $\mu[u, u'']$. Therefore, we have $\mu[u, u'] * \mu[u', u''] \subseteq \mu[u, u'']$ as expected. □

A converse link exists between the relations R_0 and R.

Proposition 4.8 *Let u and u'' be two points of T such that $u \leqslant u''$ and let Γ_1 and Γ_2 be two worlds of W_0 such that $R_0(\Gamma_1, \Gamma_2, \mu[u, u''])$, then there is an element u' of T such that $u \leqslant u' \leqslant u''$ and $\mu[u, u'] = \Gamma_1$, $\mu[u', u''] = \Gamma_2$.*

Proof: The points u and u'' are two pairs (Δ_1, Δ_2) and (Δ_1'', Δ_2'') respectively. By definition of T, we have

$$\Delta_1 * \Delta_2 \subseteq \Gamma_0^\star \quad \text{and} \quad \Delta_1'' * \Delta_2'' \subseteq \Gamma_0^\star$$

and, by definition of μ,

$$\Delta_1 * \mu[u, u''] \subseteq \Delta_1'' \quad \text{and} \quad \mu[u, u''] * \Delta_2'' \subseteq \Delta_2.$$

Consider two worlds Γ_1 and Γ_2 such that $R_0(\Gamma_1, \Gamma_2, \mu[u, u''])$,

$$\Gamma_1 * \Gamma_2 \subseteq \mu[u, u''].$$

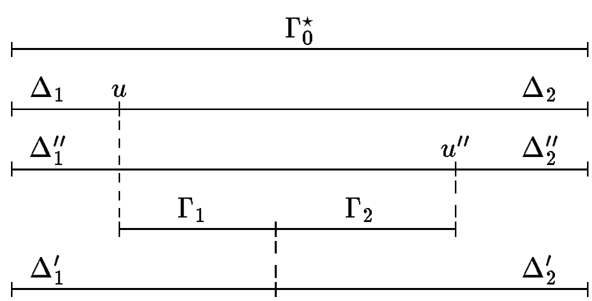
By the associativity property (proposition 4.3), there is a world Δ_1' such that

$$\Delta_1 * \Gamma_1 \subseteq \Delta_1' \quad \text{and} \quad \Delta_1' * \Gamma_2 \subseteq \Delta_1''$$

and, similarly, there is a world Δ_2' such that

$$\Gamma_1 * \Delta_2' \subseteq \Delta_2 \quad \text{and} \quad \Gamma_2 * \Delta_2'' \subseteq \Delta_2'.$$

The configuration of all these worlds can be depicted as follows:



We have to show that $u' = (\Delta_1', \Delta_2')$ is an element of T and that $u \leqslant u' \leqslant u''$. By proposition 4.6 it will follow immediately that

$$\mu[u, u'] = \Gamma_1 \quad \text{and} \quad \mu[u', u''] = \Gamma_2.$$

There exist constants b_1, b_2'', c_1, and c_2 of B such that

$$(\ell = b_1) \in \Delta_1, \quad (\ell = b_2'') \in \Delta_2'', \quad (\ell = c_1) \in \Gamma_1, \quad \text{and} \quad (\ell = c_2) \in \Gamma_2.$$

Let f and g be two sentences of Δ_1' and Δ_2' respectively. We have, on the one hand, $(f; \ell = c_2) \in \Delta_1''$ and

$$((f; \ell = c_2); \ell = b_2'') \in \Gamma_0^\star.$$

On the other hand, $(\ell = c_1; g) \in \Delta_2$ and

$$(\ell = b_1; (\ell = c_1; g)) \in \Gamma_0^\star.$$

We also have $(\ell = c_1; \ell = c_2) \in \mu[u, u'']$, $(\ell = b_1; (\ell = c_1; \ell = c_2)) \in \Delta_1''$, and

$$((\ell = b_1; (\ell = c_1; \ell = c_2)); \ell = b_2'') \in \Gamma_0^\star.$$

Using axioms A2 and L2 and the rule MONO yields

$$\begin{aligned}(f; \ell = c_2 + b_2'') &\in \Gamma_0^\star,\\ (\ell = b_1 + c_1; g) &\in \Gamma_0^\star,\\ (\ell = b_1 + c_1; \ell = c_2 + b_2'') &\in \Gamma_0^\star.\end{aligned}$$

Then, by the following theorem (cf. section 3.2.4),

T7: $(\ell = x; h_1) \wedge (h_2; \ell = y) \wedge (\ell = x; \ell = y) \Rightarrow (h_2 \wedge \ell = x; h_1 \wedge \ell = y)$,

we obtain

$$(f \wedge \ell = b_1 + c_1 \; ; \; g \wedge \ell = c_2 + b_2') \quad \in \quad \Gamma_0^\star$$

and this implies that $(f; g)$ is a sentence of $\Gamma_0^\star$. Hence, $\Delta_1' * \Delta_2' \subseteq \Gamma_0^\star$, that is, $u' = (\Delta_1', \Delta_2')$ is an element of T.

It is easily checked that $u \leqslant u' \leqslant u''$. Since $\Delta_1 * \Gamma_1 \subseteq \Delta_1'$, the sentence $(\ell = b1 + c_1)$ belongs to Δ_1'; this means that $u \leqslant u'$. Since $\Delta_1' * \Gamma_2 \subseteq \Delta_1''$, the sentence $(\ell = b_1 + (c_1 + c_2))$ belongs to Δ_1'' and this implies that $u' \leqslant u''$. □

4.3.4 Construction of $\mathcal{M}$

Using the temporal domain T and the mapping μ defined previously, we can now construct an model $\mathcal{M} = (W, R, D, I)$. $\mathcal{M}$ is obtained from the initial model $\mathcal{M}_0 = (W_0, R_0, D_0, I_0)$ as follows:

- (W, R) is the interval frame defined by T,
- the domain D is the same as D_0,
- the interpretation function I is defined by

$$I(s, [u, u']) \quad = \quad I_0(s, \mu[u, u']),$$

  for any symbol s of $\mathcal{L}$ and any interval $[u, u']$ of W.

Since the domains of $\mathcal{M}$ and $\mathcal{M}_0$ are the same, an $\mathcal{M}$-valuation v is also an $\mathcal{M}_0$-valuation. Under such a valuation, the interpretation of terms and the satisfaction of formulas of $\mathcal{L}$ in the two models are linked by the following theorem. To avoid confusion, the functions assigning values to terms in the two models are denoted $I_{[u,u']}^v$ for $\mathcal{M}$ and J_Δ^v for $\mathcal{M}_0$.

Theorem 4.9 *Let $[u, u']$ be an interval of W and let t be a term and f a formula of $\mathcal{L}$ then*

$$\begin{aligned}I_{[u,u']}^v(t) &= J_{\mu[u,u']}^v(t)\\ \mathcal{M}, [u, u'], v \models f &\text{ iff } \mathcal{M}_0, \mu[u, u'], v \models f.\end{aligned}$$

Proof: The first part is shown by an easy induction on terms. The second relation is proved by induction on formulas. The case of atomic propositions, propositional connectives and existential formulas is straightforward. Properties 4.7 and 4.8 complete the induction in the case of chop formulas. □

The following properties implies that $\mathcal{M}$ is a model of Γ_0.

Proposition 4.10 *There is an interval* $[u, u']$ *of* W *such that* $\mu[u, u'] = \Gamma_0^\star$.

Proof: By reflexivity (proposition 4.4), there are two worlds Δ_1 and Δ_2 of W_0 such that

$$\Delta_1 * \Gamma_0^\star \subseteq \Gamma_0^\star \quad \text{and} \quad \Gamma_0^\star * \Delta_2 \subseteq \Gamma_0^\star$$

with, in addition, $(\ell = 0) \in \Delta_1$ and $(\ell = 0) \in \Delta_2$. Then both $(\Delta_1, \Gamma_0^\star)$ and $(\Gamma_0^\star, \Delta_2)$ are elements of T. We can then set

$$u = (\Delta_1, \Gamma_0^\star) \quad \text{and} \quad u' = (\Gamma_0^\star, \Delta_2).$$

There is a constant b such that $(\ell = b)$ belongs to $\Gamma_0^\star$ then by D3, $(\ell = 0+b) \in \Gamma_0^\star$ this means that $u \leqslant u'$. By definition of μ, it is clear that $\Gamma_0^\star = \mu[u, u']$ (see proposition 4.6). □

$\mathcal{M}$ is then a model based on the interval frame (W, R) defined by T. By the preceding two propositions, Γ_0 is satisfied in an interval $[u, u']$ of W. It remains to show that $\mathcal{M}$ is actually an interval model. That is, we have to find a duration domain $(D, +, 0)$ and a measure m such that T, D, and m form a basis for $\mathcal{M}$ as defined in 4.1.3. This is straightforward.

The rigid symbol $+$ of $\mathcal{L}$ in $\mathcal{M}$ defines a binary operation we also denote by $+$ in D. Similarly, the interpretation of the rigid constant 0 is an element 0 of D. All the formulas D1–D5 are valid in $\mathcal{M}_0$ so by theorem 4.9 they are also valid in $\mathcal{M}$. This clearly implies that $(D, +, 0)$ is a duration domain.

The only possible definition for the measure m is to set

$$m[u, u'] \quad = \quad I(\ell, [u, u']),$$

for any interval $[u, u']$ of W. Due to the validity of L1, L2, and L3, the constraints M1–M4 are satisfied.

4.3.5 S' is complete

The two following theorems summarize the essential result of this chapter.

Theorem 4.11 *If* Γ_0 *is a consistent set of sentences with respect to* S' *then* Γ_0 *has an interval model* $\mathcal{M}$.

Proof: By completeness of S, there is an S-model $\mathcal{M}_0$ which satisfies Γ_0. An interval model $\mathcal{M}$ can be derived from $\mathcal{M}_0$ as indicated before and $\mathcal{M}$ satisfies Γ_0. □

Theorem 4.12 *If a formula f of $\mathcal{L}$ is valid in $\mathcal{K}$ then it is a theorem of S'.*

Proof: Consider a formula f which is not provable in S' and let g be the universal closure of f. As in theorem 3.17, g is not provable either. Then the set $\Gamma_0 = \{\neg g\}$ is consistent. By the preceding theorem, Γ_0 is satisfied in an interval model $\mathcal{M}$. The formula $\neg g$ is then satisfied in $\mathcal{K}$, so g and f are not valid in interval models. □

4.4 Notes

The results presented in section 4.3 rely on a particular choice for the construction of the temporal domain T from the S-model $\mathcal{M}_0$ and the state $\Gamma_0^\star$. Yet, the completeness result itself can be established similarly with different definition of T. For example, T can be chosen as the set of the elements of D smaller in the natural order than the length of $\Gamma_0^\star$. There are also different possible equivalent definition for the order on T.

However, all these various constructions rely on the fundamental propositions 4.6, 4.7 and 4.8 (with possibly minor variations). The two properties of associativity and reflexivity (propositions 4.3 and 4.4) are also essential.

In section 4.1, interval models are built from a domain D where axioms D1 to D5 are valid. This constraint can be relaxed somewhat. It is sufficient to require that D contains a subset where D1–D5 are valid, in other words D includes a duration domain. It is possible to adapt S' to this generalization of models using relativization.

For this we can introduce a new *rigid* one place predicate symbol d. Intuitively $d(x)$ can be interpreted as "x is a possible duration". Then we can replace D1–D5 with the following axioms:

D1': $d(x) \wedge d(y) \wedge d(z) \Rightarrow (x + y) + z = x + (y + z)$

D2': $d(x) \Rightarrow x + 0 = x$
$d(x) \Rightarrow 0 + x = x$

D3': $d(x) \wedge d(y) \wedge d(z) \Rightarrow (x + y = x + z \Rightarrow y = z)$
$d(x) \wedge d(y) \wedge d(z) \Rightarrow (y + x = z + x \Rightarrow y = z)$

D4': $d(x) \wedge d(y) \Rightarrow (x + y = 0 \Rightarrow x = 0 \wedge y = 0)$

D5': $d(x) \wedge d(y) \Rightarrow (\exists z)(d(z) \wedge (x + z = y \vee y + z = x))$
$d(x) \wedge d(y) \Rightarrow (\exists z)(d(z) \wedge (z + x = y \vee z + y = x))$,

and add the axioms $D0'$ which specifies that 0 is a possible duration:

D0': $d(0)$.

In the same way, the modal axiom L2 has to be modified:

L2': $d(x) \wedge d(y) \Rightarrow (\ell = x + y \Leftrightarrow (\ell = x; \ell = y))$

and we need to specify that ℓ is always a duration:

L0': $d(\ell)$.

Then the new proof system can be shown to be complete for the extended class of interval models. It suffices to adapt the construction of the order on T. The interpretation of the rigid symbol d defines a subset E of D such that $(E, +, 0)$ is a duration domain.

If Γ_0 is a consistent set of sentences with respect to S', theorem 4.11 shows that Γ_0 is satisfied by an interval model $\mathcal{M}$. This result can be refined by examining the construction of $\mathcal{M}_0$ and of $\mathcal{M}$:

- both the duration domain and the temporal domain of $\mathcal{M}$ are countable,
- the temporal domain T of $\mathcal{M}$ has a smallest element t_{min} and a largest element t_{max} and Γ_0 is satisfied in the interval $[t_{min}, t_{max}]$.

Chapter 5

Examples of applications

5.1 Extensions of S'

In this chapter, we give examples of applications and extensions of the preceding completeness results. In order to simplify the presentation, we use standard abbreviations:

- *true* denotes an arbitrary tautology,
- $(x \neq y)$ stands for $\neg(x = y)$,
- $\Diamond f$ for $((true; f); true)$ and
- $\Box f$ for $\neg\Diamond\neg f$.

Informally, $\Diamond$ and $\Box$ can be interpreted as "in some sub-interval" and "in all sub-interval" respectively (see [14]).

We will consider several extensions of S' obtained by adding new axioms. If S'' is such a proof system then S'' is *consistent* if no contradiction can be derived in S'': there is no sentence f such that

$$\vdash_{S''} f \quad \text{and} \quad \vdash_{S''} \neg f.$$

If S'' is consistent, we can consider sets of sentences which are consistent or maximal consistent with respect to S''.

Assume then S'' is consistent. In this case, any set Γ, consistent w.r.t. S'', can be extended to a set $\Gamma^\star$ maximal consistent w.r.t. S''. The set $\Gamma^\star$ is also consistent with respect to S' and by theorem 4.11 there is an interval model $\mathcal{M}$ which satisfies $\Gamma^\star$. Furthermore, this model can be obtained so that

- it is based on a countable temporal domain T,
- T has a smallest t_{min} and a largest t_{max} element,
- $\Gamma^\star$ is satisfied in $[t_{min}, t_{max}]$.

In any proof system which includes the necessity rule N, we have

$$\text{if} \quad \vdash f \quad \text{then} \quad \vdash \Box f,$$

for any formula f. This holds for S' and all its extensions, in particular for S''. By construction, all the theorems of S'' must be in $\Gamma^\star$ then for any theorem f of S'', $[t_{min}, t_{max}]$ satisfies $\Box f$. It follows easily that f is satisfied in any sub-interval $[t, t']$ of $[t_{min}, t_{max}]$, that is in any interval of the model $\mathcal{M}$.

In summary, if S'' is a consistent axiomatic system which extends S' and Γ is a set of sentences consistent w.r.t. S'', then Γ is satisfied in an interval model $\mathcal{M}$ where all the theorems of S'' are valid.

This result will be used in the sequel to show completeness of proof systems corresponding to various sub-classes of interval models. First, we consider classes of interval models based on dense temporal domains.

5.2 Axiomatizations of dense time

A temporal domain $(T, \leqslant)$ is *dense* if $\leqslant$ is a dense order on T: for any instants t and t' of T such that $t < t'$ there exists an instant u such that $t < u < t'$. We denote by $\mathcal{K}_{dense}$ the class of interval models based on dense temporal domains. The addition of a single axiom to S' provides an adequate proof system for $\mathcal{K}_{dense}$. This axiom is a modal one, similar to L1–L3. It relates the chop operator with the length of intervals.

From another point of view, it is possible to express density assumptions as first-order properties of the duration domain. Due to constraint M4 on measures and the presence of axiom L2 in the proof system, the addition on a duration domain D and the order on the associated temporal domain T are tightly related. Any interval model where the natural order on D is dense must also have a dense temporal domain.

5.2.1 Dense temporal domains

Let $\mathcal{M}$ be an interval model based on a dense temporal domain $(T, \leqslant)$. It is clear that the following sentence is valid in $\mathcal{M}$:

$$\text{L4:} \quad \ell \neq 0 \Rightarrow (\ell \neq 0 \;;\; \ell \neq 0).$$

This simply says that any non-point interval $[t, t']$ can be split into two non-point intervals $[t, u]$ and $[u, t']$. Note that the converse of L4 holds in any interval model and can be proved in S' using D4 and L2.

Let $S' + L4$ be the new proof system obtained by adding L4 to S'. This new system is sound for dense-timed interval models. This also means that $S' + L4$ is consistent. Using the preceding remark it is easy to show that $S' + L4$ is complete for $\mathcal{K}_{dense}$. If Γ is consistent w.r.t $S' + L4$ there is an interval model $\mathcal{M}$ where Γ is satisfied and where axiom L4 is valid. This implies immediately that the temporal domain T of $\mathcal{M}$ is dense and $\mathcal{M}$ is in the class $\mathcal{K}_{dense}$. By the same argument as in theorem 4.12 $S' + L4$ is complete for $\mathcal{K}_{dense}$.

5.2.2 Dense duration domains

We can add to the proof system S' the following axiom

$$\text{D6:} \quad (\forall x)(x \neq 0 \Rightarrow (\exists y)(\exists z)(x = y + z \wedge y \neq 0 \wedge z \neq 0)).$$

If a duration domain D satisfies this axiom every non null duration is the sum of two non-null durations. As a consequence, the natural order on D is a dense ordering. We say that a duration domain which satisfies D6 is *dense*. We denote by $\mathcal{K}'_{dense}$ the class of interval models based on a dense duration domain, that is where D6 is valid. As previously, $S' + D6$ is the axiomatic system obtained by adding D6 to S'.

$\mathcal{K}'_{dense}$ is a sub-class of $\mathcal{K}_{dense}$. This is a consequence of constraint M4 on measures and can be shown by deriving L4 in $S' + D6$:

1	$\ell = x + y \Rightarrow (\ell = x; \ell = y)$	L2, PC
2	$(x = 0; \ell = y) \Rightarrow x = 0$	R
3	$x \neq 0 \Rightarrow \neg(x = 0; \ell = y)$	PC, 2
4	$(\ell = x; \ell = y) \wedge \neg(x = 0; \ell = y) \Rightarrow$ $(\ell = x \wedge x \neq 0; \ell = y)$	A1
5	$\ell = x \wedge x \neq 0 \Rightarrow \ell \neq 0$	PC, Ident
6	$(\ell = x; \ell = y) \wedge x \neq 0 \Rightarrow (\ell \neq 0; \ell = y)$	Mono, PC, 3–5
7	$(\ell \neq 0; \ell = y) \wedge y \neq 0 \Rightarrow (\ell \neq 0; \ell \neq 0)$	Same as 2–6
8	$\ell = x + y \wedge x \neq 0 \wedge y \neq 0 \Rightarrow (\ell \neq 0; \ell \neq 0)$	PC, 1, 6, 7
9	$(\exists x)(\exists y)(\ell = x + y \wedge x \neq 0 \wedge y \neq 0) \Rightarrow (\ell \neq 0; \ell \neq 0)$	G, 8, PC
10	$\ell \neq 0 \Rightarrow (\exists x)(\exists y)(\ell = x + y \wedge x \neq 0 \wedge y \neq 0)$	D6, Q2
11	$\ell \neq 0 \Rightarrow (\ell \neq 0; \ell \neq 0)$	PC, 9, 10.

The use of Q2 at line 10 is permitted because the formula is chop-free.

On the other hand, D6 is not provable in $S' + L4$. It is not difficult to construct an interval model where the temporal domain is dense and the duration domain is not. For example, it suffices to consider a temporal domain T reduced to a single point and take $D = \mathbb{N}$. T is trivially dense but D is not. Hence, $\mathcal{K}'_{dense}$ is a strict sub-class of $\mathcal{K}_{dense}$.

Obviously, $S' + D6$ is complete and sound for $\mathcal{K}'_{dense}$. Any set of sentences consistent w.r.t $S' + D6$ is satisfied in a interval model where all the theorems of $S' + D6$ are valid, in particular $D6$ is valid. By definition, such a model belongs to $\mathcal{K}'_{dense}$.

More generally, various assumptions on duration domains can be considered. If these assumptions can be expressed in first order logic, they can be added as first-order axioms to D1–D5. This forms a first order theory $\mathcal{D}$ and a class $\mathcal{K}_{\mathcal{D}}$ of interval models can be associated with $\mathcal{D}$ in a natural way. An interval model $\mathcal{M}$ belongs to $\mathcal{K}_{\mathcal{D}}$ if its duration domain is a first-order model of $\mathcal{D}$ or, equivalently, if all the axioms of $\mathcal{D}$ are valid in $\mathcal{M}$.

Provided $\mathcal{D}$ is consistent as a first order theory, $\mathcal{K}_{\mathcal{D}}$ is non-empty. The proof system $S' + \mathcal{D}$ obtained by adding to S' all the new assumptions on duration domains is consistent. It is also trivially sound and complete for $\mathcal{K}_{\mathcal{D}}$.

5.3 Towards traditional ITL

5.3.1 From states to intervals

Our notion of interval model may seem a bit awkward to represent real real-time systems. A more intuitive and commonly adopted view is to introduce a notion of state which represent an instantaneous observation of a system and to specify how the state can evolve with time.

For example, assume one observes a simple system which consists of two variables X_1 and X_2 taking values in a set E. The instantaneous state of the system at an instant t is then the pair of values (x_1, x_2) of the two variables X_1 and X_2. The behaviour of the system over a period of time $[0, t]$ is completely determined by two functions:

$$\bar{X}_1 : [0,t] \to E \quad \text{and} \quad \bar{X}_2 : [0,t] \to E,$$

where $\bar{X}_j(u)$ is the value of the variable X_j at instant u.

In its traditional form [21], ITL adopts a similar point of view:

- ◇ A system is composed of a collection of variables $\{X_j \mid j \in J\}$.
- ◇ A state is an instantaneous observation of the values carried by these variables.
- ◇ An evolution of the system over a period $[0, t]$ is given by a collection of functions $\{\bar{X}_j \mid j \in J\}$ from $[0, t]$ to some set E[1].

In order to specify such systems in an interval-based formalism, traditional ITL adopts a simple semantic convention: the interpretation of a variable X_j in an interval $[u, v]$ is its value at the beginning of the interval, namely $\bar{X}_j(u)$. This is similar to [25].

5.3.2 Interval models based on states

We now consider a new class $\mathcal{K}_{states}$ of interval models which obey this semantical constraint. A simple extension of S' provides a complete and proof system for $\mathcal{K}_{states}$.

For simplicity, we assume that the state of a system is represented by a countable collection of boolean values. We consider an interval language $\mathcal{L}$

[1] In traditional ITL time is discrete and a finite sequence $s_0, \ldots, s_t$ of states is used instead of a collection of functions [21].

which includes a countable set of variables $\{X_j \mid j \in J\}$ as flexible propositional symbols. The proposition X_j are called *state variables*.

Let $\mathcal{M} = (W, R, D, I)$ be an interval model for $\mathcal{L}$ based on a temporal domain T. The above semantical convention translates to the following constraint on I: for any interval $[t, t']$ and any state variable X_j,

$$I(X_j, [t, t']) \quad = \quad I(X_j, [t, t]). \tag{5.1}$$

With such a constraint, the function $\bar{X}_j$ representing the evolution of the variable X_j can be simply defined by

$$\bar{X}_j(t) \quad = \quad I(X_j, [t, t]).$$

In other word, we have identified the instant t with the point interval $[t, t]$.

We call state-based model any interval model $\mathcal{M}$ which satisfies constraint 5.1 and we denote by $\mathcal{K}_{states}$ the class of state-based models.

5.3.3 Associated proof system

A new proof system for $\mathcal{K}_{states}$ is obtained by adding to S' the following axioms:

$$\text{A3:} \quad \begin{array}{rcl} (X_j; true) & \Rightarrow & X_j \\ (\neg X_j; true) & \Rightarrow & \neg X_j \end{array}$$

for every state variable X_j. These new axioms allow us to derive various theorems. For example, the two following ones

$$X_j \Leftrightarrow (X_j \wedge \ell = 0; true) \quad \text{and} \quad \neg X_j \Leftrightarrow (\neg X_j \wedge \ell = 0; true)$$

which correspond directly to constraint 5.1.

Before deriving these formulas, we first show that the sentence $(\ell = 0; true)$ is a theorem of S':

1	$\ell = x \Rightarrow \ell = 0 + x$	PC, D2
2	$\ell = 0 + x \Rightarrow (\ell = 0; \ell = x)$	L2
3	$\ell = x \Rightarrow (\ell = 0; true)$	Mono, PC, 1, 2
4	$(\forall x)(\ell = x \Rightarrow (\ell = 0; true))$	G, 3
5	$(\exists x)(\ell = x) \Rightarrow (\ell = 0; true)$	PC, 4
6	$(\exists x)(\ell = x)$	Ident, PC
7	$(\ell = 0; true)$	MP, 5, 6

We can use this theorem to derive the equivalence $X_j \Leftrightarrow (X_j \wedge \ell = 0; true)$:

8	$X_j \wedge \ell = 0 \Rightarrow X_j$	Tauto
9	$(X_j \wedge \ell = 0; true) \Rightarrow (X_j; true)$	Mono, 8
10	$(X_j; true) \Rightarrow X_j$	A3
11	$(X_j \wedge \ell = 0; true) \Rightarrow X_j$	PC, 9, 10
12	$(\neg X_j; true) \Rightarrow \neg X_j$	A3
13	$X_j \Rightarrow \neg(\neg X_j; true)$	PC, 12
14	$\neg(\neg X_j; true) \wedge (\ell = 0; true) \Rightarrow (X_j \wedge \ell = 0; true)$	A1, PC, Mono
15	$X_j \Rightarrow (X_j \wedge \ell = 0; true)$	PC, 7, 13, 14
16	$X_j \Leftrightarrow (X_j \wedge \ell = 0; true)$	PC, 11, 15.

The other equivalence can be derived in the same way, by replacing X_j with $\neg X_j$ in the proof.

We call *state formula* any formula built from state variables and propositional connectives. For example, $X_1 \wedge \neg X_2$, $X_3 \wedge X_4 \Rightarrow \neg X_1 \vee X_2$ are state formulas. By an easy induction, axiom A3 generalizes to any state formula Y:

$$\vdash_{S'+A3} (Y; true) \Rightarrow Y \quad \text{and} \quad \vdash_{S'+A3} (\neg Y; true) \Rightarrow \neg Y.$$

Re-using the same derivation as above with Y instead of X_j shows that the two following sentences are theorems of $\mathcal{K}_{states}$:

$$Y \Leftrightarrow (Y \wedge \ell = 0; true) \quad \text{and} \quad \neg Y \Leftrightarrow (\neg Y \wedge \ell = 0; true).$$

Hence, state formulas behave just like state variables and a function $\bar{Y}$ can be associated with any state formula Y in the same way as $\bar{X}_j$ is associated with the state variable X_j. If $\mathcal{M}$ is an interval model of $\mathcal{K}_{states}$ with temporal domain T then for any instant t, $\bar{Y}(t) = 1$ iff $[t, t] \models Y$.

5.3.4 Soundness and completeness

If an interval model $\mathcal{M}$ satisfies condition 5.1, then A3 is valid in $\mathcal{M}$. Indeed, if $[t, t']$ is an arbitrary interval of $\mathcal{M}$ such that

$$[t, t'] \models (X_j; true)$$

then there is a point u such that

$$t \leqslant u \leqslant t' \quad \text{and} \quad [t, u] \models X_j,$$

that is $I(X_j, [t, u]) = 1$. Then $I(X_j, [t, t]) = 1$ and $I(X_j, [t, t']) = 1$, hence

$$[t, t'] \models X_j.$$

The validity of the other half of A3 is as straightforward. Hence, $S' + A3$ is sound for the class $\mathcal{K}_{states}$.

Any set of sentences Γ consistent w.r.t. $S' + A3$ is satisfied in an interval model $\mathcal{M}$ where axiom A3 is valid. It is routine to check that condition 5.1 is satisfied by $\mathcal{M}$.

Let X_j be a state variable and $[t, t']$ an interval of $\mathcal{M}$. If $I(X_j, [t, t']) = 1$ then $[t, t']$ satisfies X_j. By the equivalence above,

$$[t, t'] \models (\ell = 0 \wedge X_j; true),$$

this implies that $I(X_j, [t, t]) = 1$. Similarly if $I(X_j, [t, t']) = 0$ then $[t, t']$ satisfies $\neg X_j$,

$$[t, t'] \models (\ell = 0 \wedge \neg X_j; true),$$

and $I(X_j, [t, t]) = 0$. We can conclude that $S' + A3$ is complete; $\mathcal{M}$ is a state-based model.

5.4 Compactness and finite variability

Another consequence of theorem 4.11 is a property analogous to the compactness theorem of first order logic [6]. As an application of this theorem we study the problem of expressing finite variability in ITL.

5.4.1 Compactness

The compactness theorem for ITL is the following.

Theorem 5.1 *Let $\mathcal{L}$ be an ITL language and Γ be a set of sentences of $\mathcal{L}$. Γ has an interval model if and only if every finite subset of Γ has an interval model.*

Proof: One direction of the theorem is obvious. If $\mathcal{M}$ is an interval model of Γ then every finite subset of Γ is satisfied in $\mathcal{M}$.

For the other direction, let $\Sigma = \{f_1, \ldots, f_n\}$ be a finite subset of Γ. Since Σ has an interval model, the conjunction $(f_1 \wedge \ldots \wedge f_n)$ is satisfiable in $\mathcal{K}$. This means that

$$\not\vdash_{S'} \neg(f_1 \wedge \ldots \wedge f_n)$$

for S' is sound for interval models. Then Γ is consistent with respect to S' and, by theorem 4.11, Γ has an interval model. □

5.4.2 Finite variability

Most formalisms proposed for modelling and reasoning about real-time systems are dedicated to *digital systems*. The temporal behaviour of such systems is not continuous but consist of a succession of discrete steps representing a change in the state of the system. Commonly, real-time formalisms assume *finite variability*[2]: only a finite number of steps can be performed within a finite period of time [8, 22, 2]. So called Zeno's behaviours [2] where a system performs an infinite sequence of steps while time advances closer and closer to a limit are then rejected.

In the duration calculus [8, 14], finite variability ensures that the concept of duration is well defined. A variant proposed in [15] achieves the same effect with a less stringent condition.

In this section, we investigate the relation between finite variability and ITL. Various strong forms of the assumption can be expressed in ITL, such has having variability n or at least n. However finite variability itself cannot be expressed in ITL. This can be shown using the compactness theorem.

Finite variability in interval models

Our starting point is the class of $\mathcal{K}_{states}$ of state-based models based on a countable set $\{X_j \mid j \in J\}$ of state variables. Syntactically X_j is a flexible proposition in an interval language $\mathcal{L}$.

[2] Terminology varies. Finite variability is called *divergence* in [16] whereas *divergence* designate systems which violate finite variability in [15].

Let $\mathcal{M}$ be a model of $\mathcal{K}_{states}$ with temporal domain T. By definition, the interpretation function I of $\mathcal{M}$ is such that,

$$I([t,t'],X_j) \quad = \quad I([t,t],X_j)$$

and we can associate with X_j a function $\bar{X}_j : T \to \{0,1\}$ defined by

$$\bar{X}_j(t) \quad = \quad I([t,t],X_j).$$

For any natural number n, we say that X_j has *variability* n in an interval $[t,t']$ of $\mathcal{M}$ if $[t,t']$ can be decomposed in $n+1$ sub-intervals where the function $\bar{X}_j$ is constant and has different values in successive intervals. In other words, the value of $\bar{X}_j$ changes exactly n times in $[t,t']$. We also say that X_j has variability at least n in $[t,t']$ if the value of $\bar{X}_j$ changes at least n times in $[t,t']$.

More formally, X_j has variability n in $[t,t']$ if there exist $n+2$ elements $t_0,\ldots,t_{n+1}$ of T such that

- $\diamond$ $t = t_0 < t_1 < \ldots < t_n < t_{n+1} = t'$,
- $\diamond$ for all i in $0,\ldots,n$, $\quad \bar{X}_j(u) = \bar{X}_j(t_i)$ if $t_i \leqslant u < t_{i+1}$.
- $\diamond$ for all i in $1,\ldots,n$, $\quad \bar{X}_j(t_{i-1}) \neq \bar{X}_j(t_i)$.

X_j has variability at least n if there are n elements $t_1,\ldots,t_n$ of T such that

- $\diamond$ $t \leqslant t_1 < t_2 < \ldots < t_n < t'$,
- $\diamond$ for all i in $1,\ldots,n-1$, $\quad \bar{X}_j(t_i) \neq \bar{X}_j(t_{i+1})$.

Previously, intervals were only considered as pairs of instants. In the above definitions, we have adopted a slightly different point of view: $[t,t']$ is interpreted as the set of instants u such that $t \leqslant u < t'$. We then say that u is in $[t,t']$ if $t \leqslant u < t'$. Point-intervals $[t,t]$ are then empty and X_j has not variability n in $[t,t]$.

A state variable X_j is said to have finite variability in $[t,t']$ if it has variability n for some natural number n. If X_j has variability at least n, then either X_j has finite variability m for $m > n$ or X_j has infinite variability in $[t,t']$.

Further distinction can be made between different forms of infinite variability (see [15]). An extreme case is where $\bar{X}_j$ "changes everywhere", for example if $\bar{X}_j$ is the the function from the real interval $[0,1]$ to $\{0,1\}$ which assigns 1 to rational numbers and 0 to irrational numbers. In other situations, X_j can have infinite variability in $[t,t']$ but finite variability in every strict prefix or suffix of $[t,t']$.

Expressing variability constraints

The properties "X_j has variability n'' and "X_j has variability at least n" can be expressed in ITL for any fixed n. There are formulas $A_n(X_j)$ and $B_n(X_j)$ such that for any interval $[t, t']$ of a model $\mathcal{M}$ of $\mathcal{K}_{states}$,

- $[t, t'] \models A_n(X_j)$ iff X_j has variability n in $[t, t']$ and
- $[t, t'] \models B_n(X_j)$ iff X_j has variability at least n in $[t, t']$.

In order to define $A_n(X_j)$ we use the following abbreviation. For an arbitrary state formula Y, we set

$$\lceil Y \rceil \;\triangleq\; \ell \neq 0 \wedge \neg(true \; ; \; \neg Y \wedge \ell \neq 0).$$

Let $\mathcal{M}$ be a model of $\mathcal{K}_{states}$ and $[t, t']$ be an interval of $\mathcal{M}$, then $[t, t']$ satisfies $\lceil Y \rceil$ if and only if $[t, t']$ is non-empty and for any u in $[t, t']$, $[u, t']$ satisfies Y. Therefore, if $[t, t']$ satisfies $\lceil Y \rceil$, $\bar{Y}(u)$ is equal to 1 (i.e. true) for any u such that $t \leqslant u < t'$.

X_j has variability 0 on $[t, t']$ if it is either true everywhere or false everywhere on $[t, t']$. This can be expressed by the formula

$$\lceil X_j \rceil \vee \lceil \neg X_j \rceil.$$

Similarly, X_j has variability 1 on $[t, t']$ if there is an instant u in $[t, t']$ such that either X_j is constantly true on $[t, u]$ and false on $[u, t']$ or, conversely, constantly false on $[t, u]$ and true and $[u, t']$. This can be formalized as

$$(\lceil X_j \rceil \; ; \; \lceil \neg X_j \rceil) \vee (\lceil \neg X_j \rceil \; ; \; \lceil X_j \rceil).$$

The formula for "X_j has variability n" is obtained in the same way as a disjunction of two chop-formulas where $\lceil \neg X_j \rceil$ and $\lceil X_j \rceil$ altern.

More precisely, the fact that X_j has variability n is expressed by the formula $A_n(X_j)$ defined by

$$A_n(X_j) \;\triangleq\; A_n^+(X_j) \vee A_n^-(X_j),$$

where $A_n^+(X_j)$ and $A_n^-(X_j)$ are constructed recursively as follows:

$$\begin{array}{rcl} A_0^+(X_j) & \triangleq & \lceil X_j \rceil \\ A_0^-(X_j) & \triangleq & \lceil \neg X_j \rceil \\ A_{n+1}^+(X_j) & \triangleq & (\lceil X_j \rceil \; ; \; A_n^-(X_j)) \\ A_{n+1}^-(X_j) & \triangleq & (\lceil \neg X_j \rceil \; ; \; A_n^+(X_j)). \end{array}$$

For expressing that X_j has variability at least n on $[t, t']$, it suffices to specify that $[t, t']$ can be divided in $n + 1$ successive intervals where X_j is alternatively true and false. For example, variability at least two is expressed by

$$(X_j \; ; \; (\neg X_j \; ; \; X_j)) \vee (\neg X_j \; ; \; (X_j \; ; \; \neg X_j)).$$

The sentences $B_n(X_j)$ are defined in the same way as $A_n(X_j)$:

$$B_n(X_j) \quad \triangleq \quad B_n^+(X_j) \vee B_n^-(X_j),$$

where $B_n^+(X_j)$ and $B_n^-(X_j)$ are constructed recursively as follows:

$$\begin{aligned} B_0^+(X_j) &\triangleq X_j \\ B_0^-(X_j) &\triangleq \neg X_j \\ B_{n+1}^+(X_j) &\triangleq (X_j \ ; \ B_n^-(X_j)) \\ B_{n+1}^-(X_j) &\triangleq (\neg X_j \ ; \ B_n^+(X_j)). \end{aligned}$$

It is clear that variability n implies variability at least n, the sentence $A_n(X_j) \Rightarrow B_n(X_j)$ is valid in $\mathcal{K}_{states}$. This can be derived using the proof system $S' + A3$ (in fact S' is sufficient).

For any state formula Y we have

$$\vdash_{S'+A3} \lceil Y \rceil \Rightarrow Y.$$

The derivation sketched below uses the fact that $(\ell = 0; true)$ is a theorem.

1	$\neg(true; \neg Y \wedge \ell \neq 0) \Rightarrow \neg(\ell = 0; \neg Y \wedge \ell \neq 0)$	PC, Mono
2	$\neg(\ell = 0; \neg Y \wedge \ell \neq 0) \wedge (\ell = 0; true) \Rightarrow (\ell = 0; Y \vee \ell = 0)$	A1, etc.
3	$(\ell = 0; Y \vee \ell = 0) \Rightarrow Y \vee \ell = 0$	L3
4	$\lceil Y \rceil \Rightarrow Y$	PC, 1-3.

Using this theorem with X_j and $\neg X_j$ for Y and the monotonicity rule yields:

$$\vdash_{S'+A3} A_n(X_j) \Rightarrow B_n(X_j).$$

Of course, we also have

$$\vdash_{S'+A3} B_m(X_j) \Rightarrow B_n(X_j),$$

provided $n \leqslant m$.

Finite variability is not expressible in ITL

Although variability n where n is fixed can be expressed in ITL, finite variability itself cannot. This is a consequence of the following proposition.

Proposition 5.2 *Let $\mathcal{L}$ be an ITL language with state variables $\{X_j \mid j \in J\}$, X_k a state variable of $\mathcal{L}$ and Γ a set of sentences of $\mathcal{L}$. If for any natural number n, there exists a state-based model $\mathcal{M}_n$ and an interval $[t, t']$ of $\mathcal{M}_n$ such that*

- *Γ is satisfied in $[t, t']$,*
- *X_k has variability m for some $m \geqslant n$,*

then there is a state-based model $\mathcal{M}$ and an interval $[t, t']$ of $\mathcal{M}$ such that

- *Γ is satisfied in $[t, t']$,*

◇ *X_k has infinite variability in $[t, t']$.*

Proof: Consider the set of sentences Γ' obtained by adding to Γ all the instances of axiom $A3$:

$$\text{A3:} \quad \begin{array}{rcl} (X_j; true) & \Rightarrow & X_j \\ (\neg X_j; true) & \Rightarrow & \neg X_j \end{array}$$

and all the sentences $B_m(X_k)$ for $m \in \mathbb{N}$.

Let Σ be a finite subset of Γ' and let n be the greatest index such that $B_n(X_k)$ belongs to Σ. By assumption there is a state-based model $\mathcal{M}_n$ and an interval $[t, t']$ of $\mathcal{M}_n$ such that Γ is satisfied in $[t, t']$, X_k has finite variability m in $[t, t']$, and $m \geqslant n$. Then,

◇ $\Sigma \subseteq \Gamma$ is satisfied in $[t, t']$,

◇ any instance of A3 is satisfied in $[t, t']$ since A3 is valid in state-based models,

◇ all the sentences of the form $B_p(X_k)$ for $p \leqslant m$ are satisfied in $[t, t']$.

As a consequence, $[t, t']$ satisfies Σ.

Using the compactness theorem 5.1, we can conclude that Γ' has an interval model $\mathcal{M}$. Since every instance of A3 is in Γ', $\mathcal{M}$ belongs to the class $\mathcal{K}_{states}$. Let $[t, t']$ be an interval of $\mathcal{M}$ which satisfies Γ'. Since all the formulas $B_n(X_k)$ are in Γ', X_k has variability at least n for arbitrary large n. Therefore X_k has infinite variability in $[t, t']$. Obviously Γ is satisfied in $[t, t']$ □

Roughly speaking, the previous proposition means that any set of sentences Γ which is satisfied by intervals where X_k has arbitrarily large finite variability is also satisfied by some interval where X_k has infinite variability. The only way to forbid infinite variability is to put a bound on the variability of X_k. This means that finite variability cannot be expressed in ITL.

The situation is somewhat similar to first-order logic. There is no set of sentences of first-order logic whose models are precisely all the finite models. Our proposition 5.2 above is the counterpart of a well-known result: if a first order theory has arbitrarily large finite models that it has an infinite model (Corollary 2.1.5, page 65 in [6]).

Chapter 6

Conclusion

In this report, we have presented two completeness results for first order interval temporal logic. These results are quite general and extend to various axiomatic systems for ITL as illustrated in chapter 5. They also allow us to prove a fundamental limitation of ITL: its inability to express finite variability.

We hope to extend the techniques developed to formal systems for the duration calculus. This requires to generalize the integration of real functions to functions defined on arbitrary (dense) temporal domain T.

The axiomatic systems S and S' are intended to be relatively close to existing proof systems presented in the literature [21, 26]. The completeness of S' for interval models delimitates the power of these proof systems. However, the construction does not guarantee completeness for the *standard* semantics of ITL or the duration calculus. These semantics are based on a particular choice of temporal domain T and of duration domain D. The techniques presented in this document do not apply easily to such cases.

However, the kind of construction developed could find some interesting applications, for example does the suppression of L1 from S provide a complete proof systems for ITL in general, that is, for the class of all possible worlds models. Also, variants of the system S' could permit to consider infinite intervals of the form $[t, \infty)$. This would enrich considerably the expressive power of ITL in particular by allowing liveness and fairness property to be specified.

Bibliography

[1] M. Abadi. The power of temporal proofs. *Theoretical Computer Science*, 65:35–83, 1989. Corrigendum in TCS 70 (1990), page 275.

[2] M. Abadi and L. Lamport. An Old-Fashioned Recipe for Real Time. Technical Report 91, Digital Equipment Corporation, System Research Center, October 1992.

[3] R. Alur, C. Courcoubetis, and D. Dill. Model-checking in dense real-time. *Information and Computation*, 104(1):2–34, May 1993.

[4] R. Alur and T. A. Henzinger. Real-time logics: Complexity and expressiveness. *Information and Computation*, 104(1):35–77, May 1993.

[5] P. B. Andrews. *An Introduction to Mathematical Logic and Type Theory: To Truth through Proof.* Academic Press, 1986.

[6] C. C. Chang and H. J. Keisler. *Model Theory.* North-Holland, 1973.

[7] Z. Chaochen, M. R. Hansen, and P. Sestoft. Decidability and undecidability results for duration calculus. In *Proc. of STACS'93*, pages 58–68. Springer-Verlag, LNCS 665, 1993.

[8] Z. Chaochen, C. A. R. Hoare, and A. P. Ravn. A calculus of durations. *Information Processing Letters*, 40(5):269–276, December 1991.

[9] E. A. Emerson. Temporal and modal logic. In *Handbook of Theoretical Computer Science*, pages 995–1072. Elsevier, 1990.

[10] E. A. Emerson, A. Mok, A. P. Sistla, and J. Srinisavan. Quantitative temporal reasoning. In *Computer-Aided Verification*, pages 136–145. Springer-Verlag, LNCS 531, 1990.

[11] J. W. Garson. Quantification in modal logic. In *D. Gabbay and F. Guenthner (eds.), Handbook of Philosophical Logic*, volume II, pages 249–307. Reidel, 1984.

[12] J. Y. Halpern and Y. Shoham. A propositional modal logic of time intervals. *Journal of the ACM*, 38(4):935–962, October 1991.

[13] A. G. Hamilton. *Logic for Mathematicians.* Cambridge University Press, 1988. Revised Edition.

[14] M. R. Hansen and Z. Chaochen. Semantics and completeness of duration calculus. In *Real-Time: Theory in Practice, REX Workshop*. Springer-Verlag, LNCS 600, 1992.

[15] M. R. Hansen, P. K. Pandya, and Z. Chaochen. Finite divergence. ProCoS II, Esprit BRA 7071 ID/DTH MRH 9/1, Technical University of Denmark, DK-2800, Lyngby, January 1994.

[16] T. A. Henzinger and P. W. Kopke. Verification methods for the Divergent Runs of Clock Systems. In *Formal Techniques in Real-Time and Fault-Tolerant Systems*, pages 351–372. Springer-Verlag, LNCS 863, September 1994.

[17] T. A. Henzinger, X. Nicollin, J. Sifakis, and S. Yovine. Symbolic model checking for real-time systems. *Information and Computation*, 111(2):193–244, 1994.

[18] G. E. Hughes and M. J. Cresswell. *An Introduction to Modal Logic*. Routledge, 1990. First published by Methuen and Co., 1968.

[19] H. R. Lewis. A logic of concrete time intervals. In *Proc. of LICS 90*, pages 380–389, 1990.

[20] B. Moszkowski. Temporal logic for multilevel reasoning about hardware. *IEEE Computer*, 18(2):10–19, February 1985.

[21] B. Moszkowski. Some very compositional temporal properties. In *Programming Concepts, Methods, and Calculi*, pages 307–326. Elsevier Science B.V. (North-Holland), 1994.

[22] X. Nicollin and J. Sifakis. An overview and synthesis on timed process algebras. In *Proc. of the third workshop on Computer Aided Verification*, 1991.

[23] B. Paech. Gentzen-systems for propositional temporal logics. In *Proc. of the 2nd Workshop on Computer Science Logic*, pages 240–253. Springer-Verlag, LNCS 385, 1988.

[24] A. P. Ravn, H. Rischel, and K. M. Hansen. Specifying and verifying requirements of real-time systems. *IEEE Trans. on Software Engineering*, 19(1):41–55, January 1993.

[25] R. Rosner and A. Pnueli. A choppy logic. In *Proc. of the IEEE Symposium on Logic in Computer Science*, pages 306–313. IEEE, 1986.

[26] J. U. Skakkebæk and N. Shankar. Towards a duration calculus proof assistant in PVS. In *Formal Techniques in Real-time and Fault-Tolerant Systems*. Springer-Verlag, LNCS 863, September 1994.

[27] Y. Venema. A modal logic for chopping intervals. *Journal of Logic and Computation*, 1(4):453–476, 1991.